VIRTUAL WAITING TIME IN SINGLE-SERVER QUEUE MODEL M|G|1 WITH RELIABLE SERVER AND CATASTROPHES

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The single-server queue model M|G|1|∞ with reliable server subject to catastrophes is considered. The transient and stationary distributions of virtual waiting time, busy period and idle state probability for the model with reliable server are obtained. Different generalizations of the basic model are considered: model with different arrival parameter service time of first and regular customers, as well as the model with non-homogeneous streams of customers and catastrophes. For those models, the virtual waiting time distribution and idle state probability are found.

Over the last two decades, an increasing interest in queueing models with "negative" customers has been observed (see reviews [1 - 3]). Different mechanisms of interaction between "regular" and "negative" customers have been considered. If an arriving "negative" customer can destroy all system workload, i.e. remove all "regular" customers which have been waiting and being served in the model this type of "negative" customer is called catastrophe or disaster. Different models with catastrophes and their applications are considered in [3-13]. Later in this paper we will use customers and catastrophes instead of "regular" and "negative" customers.

In queuing models with catastrophes subject of interest are three main characteristics: queue size, busy period, and virtual waiting time or workload of the model. One of the first queuing models with catastrophes was considered in [4]. The product form solution for queue size distribution in steady state was obtained. The queuing model M|G|1 with catastrophes was investigated in [5]. The stationary workload process is considered and the analog of Pollaczeck-Khinchin formula for the model was found. The single server model M|G|1 with "stochastic clearing" mechanism-disaster and with server repair after the disaster was considered in [6]. The distribution of number of customers in the model and sojourn time distribution in steady state were obtained. The model Mx|G|1 with recovery time after the catastrophes was investigated in [7]. The queue size and busy period distributions in steady state are obtained. A M|G|1 queue with Poisson arrival of "negative" customers, which can remove a stochastic amount of work from the system, is considered in [8]. The queue model M|G|1 with unreliable server and catastrophes was considered in [9], using supplementary variables and supplementary event methods. The transient and steady state distributions of queue size and busy period are investigated. Single server queue MAP|G|1 with MAP arrival of customers and catastrophes was considered in [10]. The model BMAP|G|1 with Batch MAP input of customers and catastrophes was considered in [11]. The stationary distributions of actual and virtual waiting times of customers were found. The queue model M|M|1 with catastrophes and two preemptive priority customers were introduced in [12]. The waiting time distributions for different priority customers are found. The model Mr|Gr|1 with priority customers and catastrophes is considered in [13]. The transient and stationary queue size and busy period distributions of the model are obtained.

In this paper, we consider transient and stationary distributions of virtual waiting time of the models M|G|1 with catastrophes and reliable server. This model generalizes the results of [4-9, 12]. As an application of the model we consider the $M_r|G_r|1|\infty$ queues with catastrophes.

Model Description. We consider a single server queue model $M|G|1|\infty$ with reliable server and catastrophes. The arrival of customers and occurrence of catastrophes are according to a Poisson distribution with parameters λ and ν, respectively. Catastrophes can remove all the customers being in the model at the moment of their occurrence. Service times of customers are i.i.d. random variables β with general distribution function (PDF) $B(x) = P(\beta < x)$, density $b(x)$, and mean value $\overline{\beta_1}$.

If the model is empty, then the occurring catastrophe disappears without any influences on the model. If the server is busy serving customers, then the catastrophe removes all customers in the model, including the one in service. After catastrophes, the model continues its work from an empty state. Customers serve in the model according to FCFS (first come – first serve) discipline. The model has unlimited waiting space. The model is empty at the initial moment $t = 0$.

Virtual Waiting Time. We will use the following notations and definitions: $\{\xi(t),\ t \ge 0\}$ - is a stochastic process (SP) which describes workload or virtual waiting time of the model at the moment t and takes values in the state space $[0,\infty)$. SP $\{\xi(t),\ t \ge 0\}$ is a homogeneous Markov process with respect to time. We will consider two different states of the SP $\xi(t)$: 0- state, where the model is free of customers and $(0,\infty)$ - where the model is busy either serving the customers. $W(x,t) = P\{\xi(t) < x\}$ - is the PDF for SP $\{\xi(t),\ t \ge 0\}$, $p_0(t) = W(0+,t) = P\{\xi(t) = 0\}$ - is an idle state probability of the model, i.e. model is free from customers at the moment t.

$\hat{W}(x,t) = W(x,t) - p_0(t) = P\{0 < \xi(t) < x\}$ - is the PDF of total workload of the model at moment t, i.e. the model is busy, at the moment t and total workload is x [14 -16]. Obviously, between $W(x,t),\ p_0(t)$, and $\hat{W}(x,t)$ take place following relations

$$W(x,t) = \begin{cases} 0, & if\ x \le 0 \\ \hat{W}(x,t) + p_0(t), & if\ x > 0, \end{cases}.$$

By using standard arguments for $p_0(t)$ and $\hat{W}(x,t)$ we derive following equations

$$\frac{d}{dt} p_0(t) = -\lambda p_0(t) + \frac{\partial}{\partial x}\hat{W}(x,t)\big|_{x=0} + v\hat{W}(\infty,t), \tag{1}$$

$$\frac{\partial}{\partial t}\hat{W}(x,t) - \frac{\partial}{\partial x}\hat{W}(x,t) = -(\lambda + v)\hat{W}(x,t) - \frac{\partial}{\partial x}\hat{W}(x,t)\big|_{x=0} + \lambda\int_0^x \hat{W}(x-y,t)dB(y) + \lambda B(x)p_0(t),$$

with initial conditions: $\lim_{x\to\infty} W(x,t) = \lim_{x\to\infty}\hat{W}(x,t) + p_0(t) = 1,\ \ and\ \ W(0+,0) = p_0(0) = 1.$

For the steady state solution of the model we have: $\hat{W}(x) = \lim_{t\to\infty} \hat{W}(x,t)$, $p_0 = \lim_{t\to\infty} p_0(t)$. From (1) in steady state for PDF of $W(x)$ we obtain

$$\frac{\partial}{\partial x} W(x) = (\lambda + v) W(x) - \lambda \int_0^x W(x-y) dB(y) - v . \tag{2}$$

Let $\tilde{A}(\theta)$ be the Laplace - Stieltjes transformation (LST) of a function $A(t)$

$$\tilde{A}(s) = \int_0^\infty e^{-st} dA(t), \ \mathrm{Re}(s) > 0.$$

Then for $\tilde{W}(s)$ we have

$$\tilde{W}(s) = \frac{sp_0 - v}{s - v - \lambda(1 - \tilde{B}(s))}, \tag{3}$$

For steady-state probabilities p_0 and p_1 can we will use the obtained in [13] results.

$$p_0 = \frac{v}{v + \lambda[1 - \tilde{\pi}_0(v)]}, \ p_1 = \frac{\lambda[1 - \tilde{\pi}_0(v)]}{v + \lambda[1 - \tilde{\pi}_0(v)]},$$

where $\tilde{\pi}_0(v)$ is a LST of PDF of busy period of the model without catastrophes.

After substitution of p_0 into (3) we obtain the LST of virtual waiting time distribution $\tilde{W}(s)$ for the corresponding models M|G|1 with catastrophes

$$\tilde{W}(s) = \frac{v[s - v - a(1 - \tilde{\pi}_0(v))]}{[s - v - \lambda(1 - \tilde{B}(s))][v + a(1 - \tilde{\pi}_0(v))]}. \tag{4}$$

Let $\bar{\omega}_1$ be a mean value of virtual waiting time, then from (4) we derive

$$\bar{\omega}_1 = \lim_{s\to 0}\left[-\tilde{W}'(s)\right] = \frac{p_0 - [1 - \lambda b_1]}{v} = \frac{p_0 - [1 - \rho]}{v}, \tag{5}$$

where ρ is a workload of the model, $\rho = \lambda b_1$.

Obviously when the arrival rate of catastrophes approaches to zero $v \to 0$ from (4) and (5) we obtain the mean value of virtual waiting time $\bar{\omega}_1$ and idle state stationary probability for the model without catastrophes $\bar{\omega}_{01} = \frac{\lambda b_2}{2(1-\rho)}$ and $p_{00} = 1 - \rho$.

Here b_k is the k-th order moment of the service time distribution $B(t)$.

To evaluate $\bar{\omega}_1$ and p_0 for small value of a parameter of catastrophes v first we define the expansion for LST of busy period of the standard model without catastrophes

$$\tilde{\pi}_0(v) = \int_0^\infty e^{-vt} d\pi_0(t) = 1 - v\pi_{01} + \frac{v^2 \pi_{02}}{2} - \frac{v^3 \pi_{03}}{3!} + O(v),$$

where $\pi_{01} = \frac{b_1}{1-\rho}$, $\pi_{02} = \frac{b_2}{(1-\rho)^3}$, $\pi_{03} = \frac{b_3}{(1-\rho)^4} + \frac{3\lambda b_2^2}{(1-\rho)^5}$.

Then for p_0 and $\bar{\omega}_1$ we obtain

$$p_0 = (1-\rho) + \frac{v}{2} \frac{\lambda b_2}{(1-\rho)} + v^2 \left(\frac{1}{6} \frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4} \frac{(\lambda b_2)^2}{(1-\rho)^3} \right) + O(v^3), \tag{6}$$

$$\bar{\omega}_1 = \frac{p_0-(1-\rho)}{v} \cong \frac{1}{2}\frac{\lambda b_2}{(1-\rho)} + v\left(\frac{1}{6}\frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4}\frac{(\lambda b_2)^2}{(1-\rho)^3}\right) + O(v^2) \quad (7)$$

The (6) and (7) can be rewritten in more convenient form

$$p_0 = p_{00} + \frac{v}{2}\frac{\lambda b_2}{(1-\rho)} + v^2\left(\frac{1}{6}\frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4}\frac{(\lambda b_2)^2}{(1-\rho)^3}\right) + O(v^3),$$

$$\bar{\omega}_1 = \bar{\omega}_{01} + v\left(\frac{1}{6}\frac{\lambda b_3}{(1-\rho)^2} + \frac{3}{4}\frac{(\lambda b_2)^2}{(1-\rho)^3}\right) + O(v^2).$$

For the LST of PDT of virtual waiting time $\tilde{W}(s,t)$ taking into account the initial conditions we derive the following differential equation

$$\frac{\partial}{\partial t}\tilde{W}(s,t) - \tilde{W}(s,t)[s - v - \lambda(1-\tilde{B}(s))] = v - sp_0(t), \quad (8)$$

and its solution

$$\tilde{W}(s,t) = e^{\varphi(s)t}\left\{\tilde{W}(s,0) - s\int_0^t e^{-\varphi(s)u} p_0(u)du + v\int_0^t e^{-\varphi(s)u}du\right\}, \quad (9)$$

where $\varphi(s) = s - v - \lambda(1-\tilde{B}(s))$.

The expressions for unknown function $p_0(t)$ we will derive by supplementary event method [17-20].

$$\tilde{p}_0(s) = \frac{s+v}{s\{s+v+\lambda(1-\tilde{\pi}_0(s+v))\}},\ p_0 = \lim_{s\to 0} s\tilde{p}_0(s) = \frac{v}{v+\lambda(1-\tilde{\pi}_0(v))}.$$

Remark. Let's suppose that customers arrive and catastrophes occur according to non-homogeneous Poisson processes with rates $\lambda(t)$ and $v(t)$, respectively. In this case the corresponding differential equation for $\tilde{W}(s,t)$ is

$$\frac{\partial}{\partial t}\tilde{W}(s,t) - \tilde{W}(s,t)[s - v(t) - \lambda(t)(1-\tilde{B}(s))] = v(t) - sp_0(t), \quad (10)$$

and its solution is

$$\tilde{W}(s,t) = e^{\int_0^t \varphi(s,y)dy}\left\{\tilde{W}(s,0) - s\int_0^t e^{\int_0^u \varphi(s,y)dy} p_0(u)du + \int_0^t e^{\int_0^u \varphi(s,y)dy} v(u)du\right\} \quad (11)$$

where $\varphi(s,y) = s - v(y) - \lambda(y)(1-\tilde{B}(s))$.

To solve the differential equation (10) we use LST method. From (10), after taking LST, for $\tilde{W}(s,\theta)$ we get following equation

$$\tilde{W}(s,\theta) = \frac{\tilde{W}(s,0) + \frac{v}{\theta} - s\tilde{p}_0(\theta)}{\theta - \varphi(s)},\ \text{where } \tilde{W}(s,0) = s\tilde{p}_0\varphi(s) - \frac{v}{\varphi(s)}.$$

Remark. Let's consider queuing model M|G|1 with catastrophes when parameter of arrival and PDF of service time of first customer which opens the busy period are λ_0 and $B_0(x)$. Parameter of arrival and PDF of service time for other customers (during one busy period of the model) are λ and $B(x)$, respectively. We suppose that both PDFs have finite first and second moments.

The corresponding differential equations for $W(x,t)$, LST $\tilde{W}(s,t)$ and its solution are

$$\frac{\partial}{\partial t}W(x,t)-\frac{\partial}{\partial x}W(x,t)=-(\lambda+v)W(x,t)+\lambda\int_0^x W(x-y,t)dB(y)+[\lambda_0(1-B_0(x))-\lambda(1-B(x))]p_0(t),$$

$$\frac{\partial}{\partial t}\tilde{W}(s,t)-\tilde{W}(s,t)\varphi(s)=v-p_0(t)[s-\lambda_0(1-\tilde{B}_0(s))+\lambda(1-\tilde{B}(s))],$$

$$\tilde{W}(s,t)=e^{\varphi(s)t}\left\{\tilde{W}(s,0)-[s-\lambda_0(1-\tilde{B}_0(s))+\lambda(1-\tilde{B}(s))]\int_0^t e^{-\varphi(s)u}p_0(u)du+v\int_0^t e^{-\varphi(s)u}du\right\}.$$

This model can be used for M|G|1 queues with unreliable server and vocations as shown in [9, 17]

Conclusion. The considered model has many applications in field of computer science, economics, and public health. For example, distributed database systems with site failure [12], cyber - attacks and viruses in computer systems and networks [9], cleaning operations in financial and storage systems [1,2].

It is the first model that considers virtual waiting time transient solutions for non-stationary streams of customers and catastrophes. The model can be used also for Mr|Gr|1 queues with priority service of customers, vocations, and set-up time.

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