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How to Construct Wavelets on Local Fields of Positive Characteristic

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Abstract—We present an algorithm for construction step wavelets on local fields of positive characteristic.

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1. INTRODUCTION

In 2004 H. Jiang, D. Li, and N. Jin [1] introduced the notion of multiresolution analysis (MRA) on local fields $F^{(s)}$ of positive characteristic p, proved some properties and constructed "Haar MRA" and corresponding "Haar wavelets". The wavelet theory developed in [2–6]. Construction of non-Haar wavelets is the a basic problem in this theory. The problem of constructing orthogonal MRA on the field $F^{(1)}$ is studied in detail in the works [7–12]. S.F. Lukomskii, A.M. Vodolazov [13, 14] considered local field $F^{(s)}$ as a vector space over the finite field $GF(p^s)$ and constructed non-Haar wavelets. In [13] the authors construct the mask $m^{(\mathbf{0})}$ and correspondent refinable function φ using some tree with zero as a root. In this case wavelets $\Psi = (\psi^{(\mathbf{l})})_{\mathbf{l}\in GF(p^s)}$ may be found from the equality $\hat{\psi}^{(\mathbf{l})} = m^{(\mathbf{l})}(\chi)\hat{\varphi}(\chi\mathbf{A}^{-1})$ where $\mathbf{A}$ is a dilation operator, $m^{(\mathbf{l})}(\chi) = m^{(\mathbf{0})}(\chi\mathbf{r}_0^{-\mathbf{l}})$, and $\mathbf{r}_k^{\mathbf{l}}$ are Rademacher functions. In the article [15], the concept of N-valid tree was introduced and an algorithm for constructing the mask $m^{(\mathbf{0})}$ and correspondent refinable function φ was indicated in the field $F^{(1)}$. In the articles [16, 17] the mask $m^{(\mathbf{0})}$ and correspondent refinable function φ were constructed using graph which is obtained from N-valid tree by adding new arcs. But in this case we cannot define "masks" $m^{(\mathbf{l})}(\chi)$ by the equation $m^{(\mathbf{l})}(\chi) = m^{(\mathbf{0})}(\chi\mathbf{r}_0^{-\mathbf{l}})$.

In this article we give an algorithm for construction of "masks" $m^{(\mathbf{l})}(\chi)$ in general case.

2. BASIC CONCEPTS

Let p be a prime number, $s \in \mathbb{N}$, $GF(p^s)$ is finite field. Local field $F^{(s)}$ of positive characteristic p is isomorphic (Kovalski–Pontryagin theorem [18]) to the set of formal power series

$$a = \sum_{i=k}^{\infty} \mathbf{a}_i t^i, \quad k \in \mathbb{Z}, \quad \mathbf{a}_i \in GF(p^s).$$

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Addition and multiplication in the field $F^{(s)}$ are defined as sum and product of such series. Therefore we will consider local field $F^{(s)}$ of positive characteristic p as the field of sequences infinite in both directions

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \quad \mathbf{a}_j \in GF(p^s)$$

which have only finite number of elements $\mathbf{a}_j$ with negative j nonequal to zero, and the operations of addition and multiplication are defined by equalities

$$a \dot{+} b = ((\mathbf{a}_i \dot{+} \mathbf{b}_i))_{i \in \mathbb{Z}}, \quad ab = \left(\sum_{i,j: i+j=l} (\mathbf{a}_i \mathbf{b}_j) \right)_{l \in \mathbb{Z}},$$

where "$\dot{+}$" and "$\cdot$" are respectively addition and multiplication in $GF(p^s)$. The norm of the element $a \in F^{(s)}$ is defined by the equality

$$||a|| = ||(\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots)|| = \left(\frac{1}{p^s} \right)^n, \quad \mathbf{a}_n \neq \mathbf{0}.$$

Therefore $F_n^{(s)} = \{a = (\mathbf{a}_j)_{j \in \mathbb{Z}} : \mathbf{a}_j \in GF(p^s);\ \mathbf{a}_j = \mathbf{0},\ \forall j < n\}$ is a ball of radius p^{-ns}. Neighborhoods $F_n^{(s)}$ are compact subgroups of the group $F^{(s)+}$. We will denote them as $F_n^{(s)+}$. They have the following properties: 1) $\dots \subset F_1^{(s)+} \subset F_0^{(s)+} \subset F_{-1}^{(s)+} \subset \dots$; 2) $F_n^{(s)+}/F_{n+1}^{(s)+} \cong GF(p^s)^+$ и $\sharp(F_n^{(s)+}/F_{n+1}^{(s)+}) = p^s$. It is noted in [13] that the field $F^{(s)}$ can be described as a linear space over $GF(p^s)$. Using this description one may define the multiplication of element $a \in F^{(s)}$ on element $\boldsymbol{\lambda} \in GF(p^s)$ coordinatewise, i.e. $\boldsymbol{\lambda} a = (\dots \mathbf{0}_{n-1}, \boldsymbol{\lambda}\mathbf{a}_n, \boldsymbol{\lambda}\mathbf{a}_{n+1}, \dots)$, and the modulus $\boldsymbol{\lambda} \in GF(p^s)$ can be defined as

$$|\boldsymbol{\lambda}| = \begin{cases} 1, & \boldsymbol{\lambda} \neq \mathbf{0}, \\ 0, & \boldsymbol{\lambda} = \mathbf{0}. \end{cases}$$

It is also proved there, that the system $g_k \in F_k^{(s)} \setminus F_{k+1}^{(s)}$ is a basis in $F^{(s)}$, i.e. any element $a \in F^{(s)}$ can be represented as $a = \sum_{k \in \mathbb{Z}} \boldsymbol{\lambda}_k g_k$, $\boldsymbol{\lambda}_k \in GF(p^s)$. From now on we will consider $g_k = (\dots, \mathbf{0}_{k-1}, (1^{(0)}, 0^{(1)}, \dots, 0^{(s-1)})_k, \mathbf{0}_{k+1}, \dots)$. In this case $\boldsymbol{\lambda}_k = \mathbf{a}_k$. Let us define the sets

$$H_0^{(s)} = \{h \in G : \ h = \mathbf{a}_{-1} g_{-1} \dot{+} \mathbf{a}_{-2} g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-s} g_{-s}\}, \quad s \in \mathbb{N}.$$
$$H_0 = \{h \in G : \ h = \mathbf{a}_{-1} g_{-1} \dot{+} \mathbf{a}_{-2} g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-s} g_{-s}, s \in \mathbb{N}\}.$$

The set H_0 is the set of shifts in $F^{(s)}$. It is an analogue of the set of nonnegative integers.

We will denote the collection of all characters of $F^{(s)+}$ as X. The set X generates a commutative group with respect to the multiplication of characters: $(\chi * \phi)(a) = \chi(a) \cdot \phi(a)$. Inverse element is defined as $\chi^{-1}(a) = \overline{\chi(a)}$, and the neutral element is $e(a) \equiv 1$.

Following [13] we define characters r_n of the group $F^{(s)+}$ in the following way. Let $x = (\dots, \mathbf{0}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots)$, $\mathbf{x}_j = (x_j^{(0)}, x_j^{(1)}, \dots, x_j^{(s-1)}) \in GF(p^s)$. The element $\mathbf{x}_j$ can be written in the form $\mathbf{x}_j = (x_{js+0}, x_{js+1}, \dots, x_{js+(s-1)})$. In this case

$$x = (\dots, 0, x_{ks+0}, x_{ks+1}, \dots, x_{ks+s-1}, x_{(k+1)s+0}, x_{(k+1)s+1}, \dots, x_{(k+1)s+s-1}, \dots)$$

and the collection of all such sequences x is Vilenkin group. Thus the equality $r_n(x) = r_{ks+l}(x) = e^{\frac{2\pi i}{p}(x_{ks+l})}$ defines Rademacher function of $F^{(s)+}$ and every character $\chi \in X$ can be described in the following way:

$$\chi = \prod_{n \in \mathbb{Z}} r_n^{a_n}, \quad a_n = \overline{0, p-1}. \tag{1}$$

The equality (1) can be rewritten as

$$\chi = \prod_{k \in \mathbb{Z}} r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}} \tag{2}$$

and let us define $r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}} = \mathbf{r}_k^{\mathbf{a}_k}$, where $\mathbf{a}_k = (a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)}) \in GF(p^s)$. Then (2) takes the form $\chi = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k}$. We will refer to $\mathbf{r}_k^{(1,0,\dots,0)} = \mathbf{r}_k$ as the Rademacher functions. By definition we set

$$(\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k} = \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}_k}, \quad \chi^{\mathbf{b}} = \left(\prod \mathbf{r}_k^{\mathbf{a}_k}\right)^{\mathbf{b}} = \prod \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}}, \quad \mathbf{a}_k, \mathbf{b}_k, \mathbf{b} \in GF(p^s).$$

It follows that if $\mathbf{x} = ((x_k^{(0)}, x_k^{(1)}, \dots x_k^{(s-1)}))_{k\in\mathbb{Z}}$ and $\mathbf{u} = (u^{(0)}, u^{(1)}, \dots, u^{(s-1)}) \in GF(p^s)$ then

$$(\mathbf{r}_k^{\mathbf{u}}, \mathbf{x}) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u^{(l)} x_k^{(l)}}.$$

In [13] the following properties of characters are proved:

1) $\mathbf{r}_k^{\mathbf{u}\dot{+}\mathbf{v}} = \mathbf{r}_k^{\mathbf{u}} \mathbf{r}_k^{\mathbf{v}}$, $\mathbf{u}, \mathbf{v} \in GF(p^s)$.

2) $(\mathbf{r}_k^{\mathbf{v}}, \mathbf{u} g_j) = 1$, $\forall k \neq j$, $\mathbf{u}, \mathbf{v} \in GF(p^s)$.

3) The set of characters of the field $F^{(s)}$ is a linear space $(X, *, \cdot^{GF(p^s)})$ over the finite field $GF(p^s)$ with multiplication being an inner operation and the power $\mathbf{u} \in GF(p^s)$being an outer operation.

4) The set of Rademacher functions $(\mathbf{r}_k)$ is a basis in the space $(X, *, \cdot^{GF(p^s)})$.

The dilation operator $\mathbf{A}$ in local field $F^{(s)}$ is defined as $\mathbf{A}x := \sum_{n=-\infty}^{+\infty} \mathbf{a}_n g_{n-1}$, where $x = \sum_{n=-\infty}^{+\infty} \mathbf{a}_n g_n \in F^{(s)}$. In the group of characters it is defined as $(\chi \mathbf{A}, x) = (\chi, \mathbf{A}x)$.

3. STEP WAVELETS

We will consider a case of scaling function φ, which generates an orthogonal MRA, being a step function. The set of step functions constant on cosets of a subgroup $F_M^{(s)}$ with the support supp$(\varphi) \subset F_{-N}^{(s)}$ will be denoted as $\mathfrak{D}_M(F_{-N}^{(s)})$, $M, N \in \mathbb{N}$. Similarly, $\mathfrak{D}_{-N}(F_M^{(s)\perp})$ is a set of step functions, constant on the cosets of a subgroup $F_{-N}^{(s)\perp}$ with the support supp$(\varphi) \subset F_M^{(s)\perp}$.

Let $\varphi \in \mathfrak{D}_M(F_{-N}^{(s)})$ generate an orthogonal MRA $\{V_n\}$, satisfies the refinement equation $\varphi(x) = \sum_{h\in H_0^{(N+1)}} \beta_h \varphi(\mathbf{A}x \dot{-} h)$ [13], which we rewrite in a frequency from

$$\hat{\varphi}(\chi) = m^{(\mathbf{0})}(\chi) \hat{\varphi}(\chi \mathbf{A}^{-1}), \tag{3}$$

where $m^{(\mathbf{0})}(\chi) = \frac{1}{p^s} \sum_{h\in H_0^{(N+1)}} \beta_h \overline{(\chi \mathbf{A}^{-1}, h)}$ is the mask of equation (3). There exist methods for constructing $m^{(\mathbf{0})}(\chi)$ and $\hat{\varphi}(\chi)$ (see e.g. [17]). We want to construct wavelets $\psi^{(\mathbf{l})}$, $\mathbf{l} \in GF(p^s)$, $\mathbf{l} \neq \mathbf{0}$ from refinable function φ. We will find these wavelets $\psi^{(\mathbf{l})}$ from the equations $\hat{\psi}^{(\mathbf{l})}(\chi) = m^{(\mathbf{l})}(\chi) \hat{\varphi}(\chi \mathbf{A}^{-1})$, and will call the functions $m^{(\mathbf{l})}(\chi)$ masks, too. It is evident that $\hat{\psi}^{(\mathbf{0})}(\chi) = \hat{\varphi}(\chi)$.

Theorem 1. *Let $m^{(\mathbf{k})}(\chi)$ ($\mathbf{k} \in GF(p^s)$) be a masks that are constant on the cosets of a subgroup $F_{-N}^{(s)\perp}$ and periodic with any period $\mathbf{r}_1^{\mathbf{a}_1} \mathbf{r}_2^{\mathbf{a}_2} \dots \mathbf{r}_\nu^{\mathbf{a}_\nu}$, $\mathbf{a}_j \in GF(p^s)$, $\nu \in \mathbb{N}$. Define wavelets $\psi^{(\mathbf{l})}$ by the equations $\hat{\psi}^{(\mathbf{l})}(\chi) = m^{(\mathbf{l})}(\chi) \hat{\varphi}(\chi \mathbf{A}^{-1})$, where $\varphi \in \mathfrak{D}_M(F_{-N}^{(s)})$ is a refinable function. The shifts system $(\psi^{(\mathbf{l})}(x \dot{-} h^{(\mathbf{l})}))$, $\mathbf{l} \in GF(p^s)$, $h^{(\mathbf{l})} \in H_0$ will be orthonormal iff for any $\mathbf{a}_{-N} \dots \mathbf{a}_{-1} \in GF(p^s)$*

$$\sum_{\mathbf{a}_0 \in GF(p^s)} m^{(\mathbf{k})}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}) \overline{m^{(\mathbf{l})}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0})} = \delta_{\mathbf{k},\mathbf{l}}. \tag{4}$$

Proof. The sufficiency. Let $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(F_M^{(s)\perp})$. Consider scalar product $(\varphi(x \dot{-} g), \psi^{\mathbf{l}}(x \dot{-} h))$, where $g, h \in H_0$:

$$(\varphi(x \dot{-} g), \psi^{(\mathbf{l})}(x \dot{-} h)) = \int_{F^{(s)}} \varphi(x \dot{-} g) \overline{\psi^{(\mathbf{l})}(x \dot{-} h)} d\mu(x) = \int_X \hat{\varphi}_{\cdot \dot{-} g}(\chi) \overline{\hat{\psi}^{(\mathbf{l})}_{\cdot \dot{-} h}(\chi)}$$

$$= \int\limits_X \hat{\varphi}(\chi)\overline{\hat{\varphi}(\chi\mathbf{A}^{-1})(\chi,g)}(\chi,h)\overline{m^{(\mathbf{l})}(\chi)}d\nu(\chi) = \int\limits_{F_M^{(s)\perp}} |\hat{\varphi}(\chi\mathbf{A}^{-1})|^2(\chi,h\dot{-}g)m^{(\mathbf{0})}(\chi)\overline{m^{(\mathbf{l})}(\chi)}d\nu(\chi)$$

$$= \left|h\dot{-}g = \tilde{h} = \mathbf{h}_{-1}g_{-1}\dot{+}\mathbf{h}_{-2}g_{-2}\dot{+}\dots\right|$$

$$= \sum_{\mathbf{a}_{-N}\dots,\mathbf{a}_0,\dots,\mathbf{a}_{M-1}} \int\limits_{F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}},\dots,\mathbf{r}_0^{\mathbf{a}_0},\dots,\mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}} |\hat{\varphi}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}\mathbf{A}^{-1})|^2(\chi,\tilde{h})d\nu(\chi)$$

$$\times\, m^{(\mathbf{0})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})\overline{m^{(\mathbf{l})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})}d\nu(\chi)$$

$$= \sum_{\mathbf{a}_{-N},\dots,\mathbf{a}_0} m^{(\mathbf{0})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})\overline{m^{(\mathbf{l})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})}$$

$$\times \sum_{\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_{M-1}} |\hat{\varphi}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N+1}}\dots\mathbf{r}_0^{\mathbf{a}_1}\dots\mathbf{r}_{M-2}^{\mathbf{a}_{M-1}})|^2 \int\limits_{F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}\dots\mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}} (\chi,\tilde{h})d\nu(\chi). \tag{5}$$

By the orthonormality criteria for the system of shifts $(\varphi(x\dot{-}h))$ of the refinable function φ $\forall\mathbf{a}_{-N},\dots,\mathbf{a}_0 \in GF(p^s)$ the following equality holds:

$$\sum_{\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_{M-1}} |\hat{\varphi}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N+1}}\dots\mathbf{r}_0^{\mathbf{a}_1}\dots\mathbf{r}_{M-2}^{\mathbf{a}_{M-1}})|^2 = 1.$$

Consider integral from (5)

$$\int\limits_{F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}\dots\mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}} (\chi,\tilde{h})d\nu(\chi) = \frac{1}{p^{sN}}\mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h})\mathbf{r}_{-N}^{\mathbf{a}_{-N}}(\tilde{h})\dots\mathbf{r}_{-1}^{\mathbf{a}_{-1}}(\tilde{h})$$

$$= \frac{1}{p^{sN}}\mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h})\prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j,\mathbf{a}_j))},$$

where $(\mathbf{h}_j,\mathbf{a}_j) = h_j^{(0)}a_j^{(0)} + \dots + h_j^{(s-1)}a_j^{(s-1)}$ is a scalar product. Let us introduce the following notation:

$$m^{(\mathbf{0})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} = m^{(\mathbf{0})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}), \quad m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} = m^{(\mathbf{l})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}).$$

Then we obtain

$$(\varphi(\cdot\dot{-}g),\psi^{(\mathbf{l})}(\cdot\dot{-}h)) = \frac{1}{p^{sN}}\mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h})\sum_{\mathbf{a}_{-N},\dots,\mathbf{a}_0} m^{(\mathbf{0})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}\overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}\prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j,\mathbf{a}_j))}$$

$$= \begin{cases} 0, & if \quad \tilde{h} \notin F_{-N}^{(s)\perp}; \\ \frac{1}{p^{sN}}\sum\limits_{\mathbf{a}_{-N},\dots,\mathbf{a}_0} m^{(\mathbf{0})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}\overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}, & if \quad \tilde{h} = 0; \\ \frac{1}{p^{sN}}\sum\limits_{\mathbf{a}_{-N},\dots,\mathbf{a}_{-1}}\prod\limits_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j,\mathbf{a}_j))}\sum\limits_{\mathbf{a}_0} m^{(\mathbf{0})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}\overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}, & if \quad \tilde{h} \neq 0, \tilde{h} \in F_{-N}^{(s)\perp}. \end{cases} \tag{6}$$

For $(\psi^{(\mathbf{k})}(x\dot{-}g),\psi^{(\mathbf{l})}(x\dot{-}h))$ we can derive similar equality:

$$(\psi^{(\mathbf{k})}(\cdot\dot{-}g),\psi^{(\mathbf{l})}(\cdot\dot{-}h)) = \frac{1}{p^{sN}}\mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h})\sum_{\mathbf{a}_{-N},\dots,\mathbf{a}_0} m^{(\mathbf{k})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}\overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}\prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j,\mathbf{a}_j))}$$

$$= \begin{cases} 0, & if \quad \tilde{h} \notin F_{-N}^{(s)\perp}; \\ \frac{1}{p^{sN}} \sum\limits_{\mathbf{a}_{-N},\dots,\mathbf{a}_0} m^{(\mathbf{k})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} \overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}, & if \quad \tilde{h} = 0; \\ \frac{1}{p^{sN}} \sum\limits_{\mathbf{a}_{-N},\dots,\mathbf{a}_{-1}} \prod\limits_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j,\mathbf{a}_j))} \sum\limits_{\mathbf{a}_0} m^{(\mathbf{k})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} \overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}, & if \quad \tilde{h} \neq 0, \tilde{h} \in F_{-N}^{(s)\perp}. \end{cases} \tag{7}$$

Thus, if masks $m^{(\mathbf{j})}$ for all $\mathbf{a}_{-N}\dots\mathbf{a}_{-1} \in GF(p^s)$ satisfy the condition

$$\sum_{\mathbf{a}_0} m^{(\mathbf{k})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} \overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}} = \delta_{\mathbf{k},\mathbf{l}},$$

then the system of shifts $(\psi^{(\mathbf{l})}(x\dot{-}h^{(\mathbf{l})}))$, $\mathbf{l} \in GF(p^s)$ is an orthonormal system.

The necessity. Let us fix $\mathbf{k},\mathbf{l} \in FG(p^s)$ and consider equalities (6), (7) as a system of linear equation with unknowns $x^{\mathbf{k},\mathbf{l}}_{\mathbf{a}_{-N}\dots\mathbf{a}_{-1}} = \sum_{\mathbf{a}_0} m^{(\mathbf{k})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} \overline{m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0}}$ and consider the matrix A of this system. It is obvious that A is a square matrix $p^{sN} \times p^{sN}$. Let us prove that its determinant is nonequal to zero.

Let us start with $N = 1$, $s = 1$. In this case

$$A = \frac{1}{p} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{\frac{2\pi i}{p}} & e^{\frac{2\pi i}{p}\cdot 2} & \dots & e^{\frac{2\pi i}{p}\cdot(p-1)} \\ 1 & e^{\frac{2\pi i}{p}\cdot 2} & e^{\frac{2\pi i}{p}\cdot 2\cdot 2} & \dots & e^{\frac{2\pi i}{p}\cdot 2\cdot(p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{\frac{2\pi i}{p}\cdot(p-1)} & e^{\frac{2\pi i}{p}\cdot(p-1)\cdot 2} & \dots & e^{\frac{2\pi i}{p}\cdot(p-1)\cdot(p-1)} \end{pmatrix} = V,$$

where V is Vandermonde matrix, which is known to have nonzero determinant. For the sake of clarity let us consider a case $N = 2$, $s = 1$. In this case the matrix A may be represented as block matrix

$$A = \frac{1}{p} \begin{pmatrix} V & V & V & \dots & V \\ V & e^{\frac{2\pi i}{p}}V & e^{\frac{2\pi i}{p}\cdot 2}V & \dots & e^{\frac{2\pi i}{p}\cdot(p-1)}V \\ V & e^{\frac{2\pi i}{p}\cdot 2}V & e^{\frac{2\pi i}{p}\cdot 2\cdot 2}V & \dots & e^{\frac{2\pi i}{p}\cdot 2\cdot(p-1)}V \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V & e^{\frac{2\pi i}{p}\cdot(p-1)}V & e^{\frac{2\pi i}{p}\cdot(p-1)\cdot 2}V & \dots & e^{\frac{2\pi i}{p}\cdot(p-1)\cdot(p-1)}V \end{pmatrix} = V \otimes V,$$

where $\otimes$ symbol corresponds to Kronecker product. By the properties of Kronecker product $\det V \otimes V = (\det V)^p (\det V)^p = (\det V)^{2p} \neq 0$. Thus, again matrix A is nonsingular.

For the case of arbitrary N, $s = 1$ matrix A can be represented as $A = V \otimes V \otimes \dots \otimes V$ N times and will again have nonzero determinant by the properties of Kronecker product.

Similarly, when N and s are both arbitrary $A = V \otimes V \otimes \dots \otimes V$ sN times. Thus, the system is nonsingular and has a unique solution, which proves the necessity. □

Theorem 1 can be reformulated in the following way: $m^{(\mathbf{k})}(\chi)$ are the masks of corresponding step compactly supported orthonormal wavelets $\psi^{(\mathbf{l})}(\chi)$ if and only if for each $\mathbf{a}_{-N}\dots\mathbf{a}_{-1} \in GF(p^s)$ matrix $M(\mathbf{a}_{-N}\dots\mathbf{a}_{-1})$ with elements $M_{\mathbf{l},\mathbf{a}_0}(\mathbf{a}_{-N},\dots,\mathbf{a}_{-1}) = m^{(\mathbf{l})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})$ is unitary. The sufficiency of this theorem was proved in [1] (theorem 3). For step refinable functions the condition (4) is necessary and sufficient. If the condition (4) is fulfilled then the functions $\hat{\psi}^{(\mathbf{l})}(\chi) = m^{(\mathbf{l})}(\chi)\hat{\varphi}(\chi\mathbf{A}^{-1})$ form a wavelet system [1]. For a step refinable function we can describe an algorithm for constructing masks $m^{(\mathbf{l})}$ and wavelets $\psi^{(\mathbf{l})}$, $\mathbf{l} \in GF(p^s)$.

Let us assume we have all the values of $m^{(\mathbf{0})}(\chi)$. We may obtain them using an algorithm presented in [17]. Recall the notation:

$$m^{(\mathbf{0})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} = m^{(\mathbf{0})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}), \;\; m^{(\mathbf{l})}_{\mathbf{a}_{-N}\dots\mathbf{a}_0} = m^{(\mathbf{l})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}).$$

1) For each $\mathbf{a}_{-N}\ldots\mathbf{a}_{-1}$ we construct a matrix $M(\mathbf{a}_{-N}\ldots\mathbf{a}_{-1})\in Mat_{p^s\times p^s}(\mathbb{C})$ with elements $M_{\mathbf{l},\mathbf{a}_0}(\mathbf{a}_{-N}\ldots\mathbf{a}_{-1})$ the following way. The first row consists of all the values

$$m^{(\mathbf{0})}_{\mathbf{a}_{-N}\ldots\mathbf{a}_{-1},0}, m^{(\mathbf{0})}_{\mathbf{a}_{-N}\ldots\mathbf{a}_{-1},1},\ldots, m^{(\mathbf{0})}_{\mathbf{a}_{-N}\ldots\mathbf{a}_{-1},p^s-1},$$

where $\mathbf{a}_{-N}\ldots\mathbf{a}_{-1}$ are fixed and $j=a_0^{(0)}+a_0^{(1)}p+\cdots+a_0^{(s-1)}p^{s-1}$ calculated from $\mathbf{a_0}=(a_0^{(0)},a_0^{(1)},\ldots,a_0^{(s-1)})$. Supplement this matrix to unitary in the following way.

If $m^{(\mathbf{0})}_{\mathbf{a}_{-N}\ldots\mathbf{a}_{-1},0}\neq 0$ then we make $M_{\mathbf{l},\mathbf{l}}=1$ for $\mathbf{l}\neq\mathbf{0}$ and $M_{\mathbf{l},\mathbf{a_0}}=0$ for $\mathbf{l}\neq\mathbf{0},\mathbf{l}\neq\mathbf{a}_0$.

If $m^{(\mathbf{0})}_{\mathbf{a}_{-N}\ldots\mathbf{a}_{-1},0}=0$ then there exists number

$$j=j(\mathbf{a}_0)=a_0^{(0)}+a_0^{(1)}p+\cdots+a_0^{(s-1)}p^{s-1}$$

for which $m^{(\mathbf{0})}_{\mathbf{a}_{-N}\ldots\mathbf{a}_{-1},j}\neq 0$. This nonzero value exists by the property of $m^{(\mathbf{0})}$ (see e.g.[1]). In this case we make $M_{\mathbf{j},\mathbf{0}}=1$, $M_{\mathbf{l},\mathbf{l}}=1$ for $\mathbf{l}\neq\mathbf{0},\mathbf{l}\neq\mathbf{j}$, and $M_{\mathbf{l},\mathbf{a_0}}=0$ in another case.

2) Run the Gram–Schmidt process on each matrix in order to make them unitary.

3) Now for each $\mathbf{l}\in GF(p^s)$, $\mathbf{l}\neq\mathbf{0}$ we find the values of the mask $m^{(\mathbf{l})}$ from the equalities $m^{(\mathbf{l})}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\ldots\mathbf{r}_0^{\mathbf{a}_0})=M_{\mathbf{l},\mathbf{a_0}}(\mathbf{a}_{-N}\ldots\mathbf{a}_{-1})$.

4) The wavelets $\psi^{(\mathbf{l})}$ can be obtained using the formula $\hat{\psi}^{(\mathbf{l})}(\chi)=m^{(\mathbf{l})}(\chi)\hat{\varphi}(\chi\mathbf{A}^{-1})$ and performing inverse Fourier transform.

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REFERENCES

1. H. Jiang, D. Li, and N. Jin, "Multiresolution analysis on local fields," J. Math. Anal. Appl. **294**, 523–532 (2004).
2. B. Behera and Q. Jahan, "Biorthogonal wavelets on local fields of positive characteristic," Commun. Math. Anal. **15**, 52–75 (2013).
3. B. Behera and Q. Jahan, "Characterization of wavelets and MRA wavelets on local fields of positive characteristic," Collect. Math. **66**, 33–53 (2015).
4. B. Behera and Q. Jahan, "Multiresolution analysis on local fields and characterization of scaling functions," Adv. Pure Appl. Math. **3**, 181–202 (2012).
5. B. Behera and Q. Jahan, "Wavelet packets and wavelet frame packets on local fields of positive characteristic," J. Math. Anal. Appl. **395**, 1–14 (2012).
6. D. Li and H. Jiang, "The necessary condition and sufficient conditions for wavelet frame on local fields," J. Math. Anal. Appl. **345**, 500–510 (2008).
7. Yu. A. Farkov, "Multiresolution analysis and wavelets on Vilenkin groups," Facta Univ. Ser. Elec. Energ. **21**, 309–325 (2008).
8. Yu. A. Farkov, "Orthogonal wavelets on direct products of cyclic groups," Mat. Zamet. **82**, 934–952 (2007).
9. Yu. A. Farkov, "Orthogonal wavelets with compact support on locally compact abelian groups," Izv.: Math. **69**, 623–650 (2005)
10. S. Lukomskii, "Step refinable functions and orthogonal MRA on Vilenkin groups," J. Fourier Anal. Appl. **20**, 42–65 (2014).
11. V. Protasov, "Approximation by dyadic wavelets," Sb.: Math. **198**, 1665–1681 (2007).
12. V. Protasov and Yu. Farkov, "Dyadic wavelets and refinable functions on a half-line," Sb.: Math. **197**, 1529–1556 (2006).

13. S. Lukomskii and A. Vodolazov, “Non-Haar MRA on local fields of positive characteristic,” J. Math. Anal. Appl. **433**, 1415–1440 (2016).

14. A. Vodolazov and S. Lukomskii, “MRA on local fields of positive characteristic,” Izv. Sarat. Univ., Ser. Mat. Mekh. Inform. **14**, 511–518 (2014).

15. S. Lukomskii and G. Berdnikov, “N-valid trees in wavelet theory on Vilenkin groups,” Int. J. Wavelets Multiresol. Inf. Process. **13**, 1550037 (2015).

16. S. Lukomskii, G. Berdnikov, and Iu. Kruss, “On the orthogonality of a system of shifts of the scaling function on Vilenkin groups,” Math. Notes **98**, 339–342 (2015).

17. G. Berdnikov, Yu. Kruss, and S. Lukomskii, “On orthogonal systems of shifts of scaling function on local fields of positive characteristic,” Turk. J. Math. **41**, 244–253 (2017). doi 10.3906/mat-1504-7

18. I. Gelfand, M. Graev, and I. Piatetski-Shapiro, *Theory of Representations and Authomorphic Functions* (Nauka, Moscow, 1966) [in Russian].