ANALYTIC FUNCTIONS WITH SMOOTH ABSOLUTE VALUE OF BOUNDARY DATA

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Abstract. Let $f$ be an analytic function in the unit circle $D$ continuous up to its boundary $Γ$, $f(z) \neq 0$, $z \in D$. Assume that on $Γ$, the function $f$ has a modulus of continuity $ω(|f|, δ)$. In the paper we establish the estimate $ω(f, δ) \leq Aω(|f|, \sqrt{δ})$, where $A$ is some non-negative number, and we prove that this estimate is sharp. Moreover, in the paper we establish a multi-dimensional analogue of the mentioned result. In the proof of the main theorem, an essential role is played by a theorem of Hardy-Littlewood type on Hölder classes of the functions analytic in the unit circle.

Keywords: analytic function, modulus of continuity, factorization, outer function.

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INTRODUCTION

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be a bounded unit circle in the complex plane $\mathbb{C}$ and $Γ$ be its boundary. By $C_A$ we denote the set of all functions $f$ analytic in $D$ and continuous in $D \cup Γ$. If $f \in C(Γ)$, then by the symbol $ω(f, δ)$ we denote the modulus of continuity of the function $f$ on $Γ$, that is,

$$ω(f, δ) = \{\sup_{|t| \leq δ} |f(γ) − f(γe^{it})|, \quad γ ∈ Γ, \quad t ∈ R\}.$$ 

In the paper we consider the following problem: let $f \in C_A$ and at the unit circle the function $|f(e^{iθ})|$ has the modulus of continuity $ω(|f|, δ)$. What is the modulus of continuity of the function $f$ on $Γ$ and hence, on $D \cup Γ$?

Such problem in the classes of continuous functions with the modulus of continuity satisfying Bari-Stechkin condition

$$\int_0^δ \frac{ω(|f|, δ)}{t} dt + \int_δ^π \frac{ω(|f|, δ)}{t^2} dt = O(ω(|f|, δ)), \quad δ \to 0,$$

was solved first in work by V.P. Khaving and the author, see [5].

It was established that if $ω(|f|, δ)$ satisfies Bari-Stechkin condition (1) and $f(z) \neq 0$, $z \in D$, then

$$ω(f, δ) = O(ω(|f|, \sqrt{δ})), \quad δ \to 0.$$ 

Moreover, it was shown by simple examples that the obtained estimate was sharp and the condition $f(z) \neq 0$, $z \in D$, is necessary in the known sense. The detailed proof of these statements was exposed in [7]. This work gave rise to rather interesting studies in this direction. First V.P. Khavin, see [6], proposed an interesting approach for obtaining such estimates; this was done by applying the methods of the theory of singular integral operators. Later N.A. Shirokov (see [8], [10]) extended the results of such type for external functions and Hölder

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classes of order $\alpha, \alpha \in (0, +\infty)$ and he obtained the necessary and sufficient condition for $|f(e^{i\theta})|$ ensuring that the function $f$ has a prescribed modulus of continuity on the set $D \cup \Gamma$. In these works there were introduced a new characteristics and in terms of this characteristics, N.A. Shirokov obtained the results of such kind also for the Besov classes of analytic functions in $D \cup \Gamma$. And finally, we mention work [2], where it was established that this phenomenon has a local character, that is, if the modulus of continuity $|D|$ in $\mathbb{N}A$. Shirokov obtained the results of such kind also for the Besov classes of analytic functions in $D \cup \Gamma$. In these works there were introduced a new characteristics and in terms of this characteristics, and this estimate is sharp.

We note that the proof by V.P. Khavni and the proof of the results in works [2], [6], [8], [10] are based on gentle and fine methods of complex and harmonic analysis. In our opinion, the approach applied in works [5] and [7] and is based on classical theorems of Hardy-Littlewood type theorems (see [3], [4]) is more simple. In this work by developing the methods of works [5], [7] we prove such results for the modulus of continuity type theorems (see [3], [4]) is more simple. In this work by developing the methods of works [5], [7] we prove such results for the modulus of continuity $\omega(|f|, \delta)$ satisfying the classical Zygmund condition

$$\int_0^\delta \frac{\omega(|f|, \delta)}{t} dt = O(\omega(|f|, \delta)), \quad \delta \to 0. \quad (2)$$

The following statement is true.

**Theorem 1.** Let $f$ be a function in the class $C_A$ and $f(z) \neq 0$, $z \in D$. If the modulus of the continuity $\omega(|f|, \delta)$ of the function $|f|$ on $\Gamma$ satisfies Zygmund condition (2), then

$$\omega(f, \delta) = O\left(\omega(|f|, \sqrt{\delta})\right), \quad \delta \to 0,$$  

and this estimate is sharp.

**Remark 1.** A simple example, the function

$$f(z) = (1 - z)^{2\alpha} \exp\left(-\frac{1 + z}{1 - z}\right), \quad z \in D \cup \Gamma, \quad \alpha \in (0, +\infty),$$

where the principal branch of the power function is chosen, shows the sharpness of the statement of Theorem 1.

The analogue of Theorem 1 is true for analytic functions in the unit ball of the space $\mathbb{C}^n$. In order to formulate it, we introduce some notations. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\|z\| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$. We define $B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ and $S_n = \{z \in \mathbb{C}^n : \|z\| = 1\}$.

By $H(B_n)$ we denote the set of all analytic functions in $B_n$. Let $f \in H(B_n)$ and $f(z) = \sum_{k=0}^{+\infty} f_k(z)$ be the expansions of the function $f$ into homogeneous polynomials; by $R(f)$ the radial derivative of the function $f$ [11], that is,

$$R(f)(z) = \sum_{k=1}^{+\infty} kf_k(z), \quad z \in B_n.$$

We introduce also the notation:

$$C_A(B_n) = H(B_n) \cap C(B_n \cup S_n).$$

The following estimate of Theorem 1 is true in the classes $C_A(B_n)$.

**Theorem 2.** Let $f \in C_A(B_n)$ and the modulus of continuity $\omega(|f|, \delta)$ of the function $|f|$ on $S_n$ satisfies Zygmund condition (2). Then the modulus of continuity of the function on the set $B_n \cup S_n$ satisfies the estimate

$$\omega(f, \delta) \leq A\omega(|f|, \sqrt{\delta}), \quad 0 \leq \delta \leq 2.$$
Remark 2. For Hölder classes, that is, as $\omega(f, t) = t^\alpha, 0 < \alpha \leq 1, t \in [0, 2]$, the analogue of Theorem 2 was established in work [9] by N.A. Shirokov.

1. Proof of auxiliary statements

Let $f$ and $g$ be real-valued functions with a common domain $E \subset \mathbb{C}$. Then the relation $f \lesssim g$ on $E$ is equivalent to the following: there exists a positive number $A$ such that $f(z) \leq Ag(z)$ for each $z \in E$. If $f \lesssim g$ and simultaneously $g \lesssim f$, then $f(z) \approx g(z)$.

In what follows, as a function of modulus of continuity type we call a non-negative non-decreasing function $\omega$ on $[0, +\infty)$ such that $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$, $\omega(\lambda \delta) \leq 2\lambda \omega(\delta)$, $\lambda, \delta \in [0, +\infty)$.

Lemma 1. Let $\omega$ be a function of modulus of continuity type satisfying Zygmun condition (2), then

$$\omega(\delta) \ln \frac{1}{\delta} \lesssim \omega(\sqrt{\delta}), \quad \delta > 0. \quad (4)$$

Proof. By the definition we have

$$\int_{0}^{\sqrt{\delta}} \frac{\omega(u)}{u} du \leq A \omega(\sqrt{\delta}).$$

It is clear that if $1 \leq \delta$, then estimate (4) is obvious and this is why we assume that $0 < \delta < 1$. Then

$$\int_{0}^{\sqrt{\delta}} \frac{\omega(u)}{u} du = \int_{0}^{\delta} \frac{\omega(u)}{u} du + \int_{\delta}^{\sqrt{\delta}} \frac{\omega(u)}{u} du.$$

Hence,

$$\int_{0}^{\sqrt{\delta}} \frac{\omega(u)}{u} du \geq \omega(\delta) \int_{0}^{\sqrt{\delta}} \frac{du}{u} = \omega(\delta) \ln \frac{1}{\delta}.$$

It remains to employ Zygmund condition. The proof is complete. \hfill \Box

Lemma 2. Let $f \in C_A$, $t = |t|\tau$, $t \in D$, $\tau \in \Gamma$. Then the estimate

$$|f(t)| \lesssim \left( |f(\tau)| + \omega(f, (1 - |t|)) \ln \frac{1}{1 - |t|} \right)$$

is true.

Proof. We have

$$f(t) = \frac{1}{2\pi} \int_{\Gamma} P_t(\xi) f(\xi) d\xi,$$

where $P_t(\xi)$ is the Poisson kernel. This is why

$$|f(t)| \leq \frac{1}{2\pi} \int_{\Gamma} P_t(\xi) |f(\xi) - f(\tau)| d\xi + |f(\tau)|.$$

Therefore,

$$|f(t)| \lesssim (|f(\tau)| + J_\omega), \quad (5)$$

where

$$J_\omega := \frac{1}{2\pi} \int_{\Gamma} P_t(\xi) \omega(f, |\xi - \tau|) d\tau.$$
We proceed to estimating the latter integral. It is clear that
\[
J_\omega \lesssim \int_\Gamma \frac{(1 - |t|^2)\omega(f, |\xi - \tau|)}{(1 - |t|^2 + |\xi - \tau|^2)^2} |d\xi| \lesssim \int_0^\pi \frac{(1 - |t|^2)\omega(f, u)}{(1 - |t|^2 + u^2)^2} du.
\]

We define \(\omega(f, \delta)\) on \(\mathbb{R}^+ = [0, +\infty)\) as a function of modulus of continuity type (see [1], [4]). Then
\[
J_\omega \lesssim \int_0^{\pi - |t|} \omega(f, v(1 - |t|)) \frac{v}{1 + v^2} dv.
\]

Representing this integral as the sum
\[
\int_0^1 \omega(f, v(1 - |t|)) \frac{v}{1 + v^2} dv + \int_1^{\pi - |t|} \omega(f, v(1 - |t|)) \frac{v}{1 + v^2} dv,
\]
and taking into consideration that \(\frac{\omega(f, \delta)}{\delta}\) does not increase (see [1], [4]), we obtain
\[
J_\omega \lesssim \omega(f, 1 - |t|) \int_1^{\pi - |t|} \frac{v}{1 + v^2}.
\]

Hence,
\[
J_\omega \lesssim \omega(f, 1 - |t|) \ln \frac{1}{1 - |t|}.
\]

By (5), (6) we arrive at the statement of the lemma.

\begin{lemma}
(see [3]). Let \(\lambda\) be a positive non-decreasing function on \((0, 1)\) and
\[
\int_0^1 \lambda(u) du < +\infty.
\]
Assume that \(f\) is an analytic function in \(D\) such that
\[
\sup_{z \in D} \left\{ \frac{|f'(z)|}{\lambda(|z|)} \right\} < +\infty.
\]
Then the function \(f\) belongs to the class \(C_A\) and
\[
\omega(f, \delta) \lesssim \int_{1 - \delta}^1 \lambda(u) du.
\]
\end{lemma}

\begin{lemma}
Let \(f \in C_A, f(z) \neq 0, z \in D, |f(z)| \leq 1, z \in D\). Then there exists a number \(M > 0\) possessing the following properties: for an arbitrary \(0 < a < 1\), the function \(f\) in \(D\) can be represented as
\[
f(z) = \Phi_a(z) \Psi_a(z), z \in D,
\]
where
\[
\Phi_a(z) = \exp \int_a^1 \frac{\lambda(u) du}{1 - |t|^2},
\]
and
\[
\Psi_a(z) = \exp \int_0^a \frac{\lambda(u) du}{1 - |t|^2}.
\]
\end{lemma}
where \( \Psi_a \) is an analytic function in \( D \) such that \( |\Psi_a| \) is continuously extended to the entire closed circle \( D \cup \Gamma \),
\[
a \leq |\Psi_a(t)| \leq 1 \quad \text{for each } t \in D,
\]
\[
|\Psi_a(t') - |\Psi_a(t'')| | \leq |f(t') - f(t'')|, \quad t', t'' \in \Gamma,
\]
\[
\int_{\Gamma} |\ln |\Psi_a(t)|| |dt| \leq M, \tag{7}
\]
\[
\Phi_a(z) = \exp \left( - \int_{\Gamma} S_z(t) d\mu_a(t) \right), \quad z \in D,
\]
where \( S_z(t) \) is the Schwarz kernel for the circle \( D \), \( \mu_a \) is a non-negative Borel measure \( \Gamma \), whose total variation does not exceed \( M \).

**Proof.** Let \( H_a(t) = \max(a, |f(t)|) \), \( t \in \Gamma \). We have
\[
\Psi_a(z) := \exp \left( \frac{1}{2\pi} \int_{\Gamma} S_z(t) \ln H_a(t) |dt| \right),
\]
\[
\Phi_a(z) = f(z) (\Psi_a(z))^{-1} := \exp \left( - \int_{\Gamma} S_z(t) d\mu_a(t) \right), \quad z \in D,
\]
where
\[
\mu_a(E) = - \int_{E_a} \frac{1}{a} \ln \left( \frac{|f(t)|}{a} \right) |dt| + \mu(E), \tag{8}
\]
\( \mu \) is a non-negative measure defining the singular part of the function \( f \), \( E \) is an arbitrary Borel set in \( \Gamma \), \( E_a = \{ \gamma \in \Gamma : |f(\gamma)| \leq a \} \). It is clear that \( |\Psi_a| \) on \( \Gamma \) coincides with \( H_a \). Estimate (7) is obtained by the following inequality
\[
\int_{\Gamma} |\ln |\Psi_a(\xi)|| |d\xi| = \int_{E(|f| \geq a)} |\ln |\Psi_a(\xi)|| |d\xi| + \int_{E(|f| < a)} |\ln |\Psi_a(\xi)|| |d\xi|
\]
\[
= \int_{E(|f| \geq a)} |\ln |\Psi_a(\xi)|| |d\xi| + \sigma(E_a) \ln \frac{1}{a},
\]
where \( \sigma \) is the Lebesgue measure on \( E_a \).

It remains to note that
\[
E_a = E(|f| \leq a) = E \left( \ln \frac{1}{|f|} \geq \ln \frac{1}{a} \right);
\]
re recall that \( \max(|f| \leq 1, 0 < a < 1 \).

The finiteness of the integral \( \int_{\Gamma} |\ln |f(\xi)|| |d\xi| \) implies that
\[
\sup_{A \geq 0} A \sigma(\gamma \in \Gamma : |\ln |f(\gamma)|| \geq A) < +\infty.
\]
This proves (7).

Now we are going to estimate \( \mu_a(E) \). In order to do it, we denote by \( V_a(E) \), the first term in the right hand side in (8) and we note that \( \mu_a(\Gamma) \leq V_a(\Gamma) + \mu(\Gamma) \). This is why
\[
V_a(\Gamma) \leq \int_{\Gamma} |\ln |f|| |dt| + |\ln a| \sigma(\Gamma_a),
\]
where

\[ \Gamma_a = \{ \gamma \in \Gamma : |\ln |f(\gamma)|| \geq |\ln a| \}. \]

Following the lines of the proof of inequality (7), we obtain the last statement of the lemma. The proof is complete.

**Remark 3.** Employing the Jensen inequality, it is easy to observe that if \( \|f\|_{CA} < 1 \), then

\[ |\mu_a(\Gamma)| \lesssim \ln \frac{1}{\|f(0)\|}, \quad \int_{\Gamma} |\ln |\Psi_a|| \, dt \lesssim \ln \frac{1}{\|f(0)\|}. \]

2. **Proof of theorems**

**Proof of Theorem 1.** Without loss of generality we assume that

\[ |f(t)| \leq 1, \quad t \in \Gamma. \]

Moreover, for the sake of convenience we denote \( \omega(\delta) := \omega(|f|, \delta), 0 \leq \delta \leq 2 \), and at that,

\[ |f(t')| - |f(t'')| \leq \frac{1}{2} \omega(|t' - t''|) \quad \text{for all } t', t'' \in \Gamma. \]

Employing Lemma 3, it is sufficient to establish the estimate

\[ |f'(t)| \lesssim \frac{\omega(\sqrt{1 - |t|})}{1 - |t|}, \quad t \in D. \]

Let \( t \in D \) be a fixed point in the circle \( D \) and in Lemma 4 we choose \( a = \omega(\sqrt{1 - |t|}) \).

We introduce the notations

\[ F_t(t) = \Psi_{\omega(\sqrt{1 - |t|})}(t), \quad f_t(t) = \Phi_{\omega(\sqrt{1 - |t|})}(t). \]

We observe that

\[ f'(t) = f'_t(t)F_t(t) + f''_t(t)F_t(t) \]

1°. **Estimate for \( |f_t(t)||F_t(t)| \).** To estimate this product, let us first prove the inequality

\[ |F_t(t)| \lesssim |F_t(\tau)|, \quad t = |t|\tau, \quad \tau \in \Gamma. \] (9)

By Lemma 2,

\[ |F_t(t)| \lesssim \left( |F_t(\tau)| + \omega(1 - |t|) \ln \frac{1}{1 - |t|} \right). \]

This is why to prove inequality (9), it is sufficient to establish the estimate

\[ \sup_{t \in D} \{\omega(1 - |t|) \ln \frac{1}{1 - |t|} |F_t^{-1}(\tau)| \} < +\infty. \]

Suppose first that

\[ \max(|f(\tau)|, \omega(\sqrt{1 - |t|})) = |f(\tau)|, \]

then by Lemma 4 we have

\[ |F_t(\tau)| \geq \omega(\sqrt{1 - |t|}). \]

This is why, taking into consideration the estimate

\[ \omega(1 - |t|) \ln \frac{1}{1 - |t|} \leq \omega(\sqrt{1 - |t|}), \]

we obtain

\[ \frac{\omega(1 - |t|) \ln \frac{1}{1 - |t|}}{|F_t(\tau)|} \leq \frac{\omega(\sqrt{1 - |t|})}{\omega(\sqrt{1 - |t|})} \lesssim 1. \]

Now we consider the case \( |f(\tau)| \leq \omega(\sqrt{1 - |t|}) \). Applying Lemma 1 once again, we get desired estimate (9).
We proceed to estimating the functions $|f_t(t)|$, $|F_t'(t)|$. Let
\[ \Gamma_1 = \{ \gamma \in \Gamma : \omega(|\gamma - \tau|) \leq |F_t(\tau)| \}, \quad \Gamma_2 = \Gamma \setminus \Gamma_1. \]

Then we have
\[ |f_t(t)| |F_t'(t)| \leq |F_t'(t)|. \]

At that,
\[ |F_t'(t)| = |F_t(t)| \int_\Gamma |F_t(\gamma)| \frac{2\gamma}{(\gamma - t)^2} |d\gamma| = |F_t(t)| \int_\Gamma \frac{|\ln|F_t(\gamma)| - \ln|F_t(\tau)||}{(\gamma - t)^2} |d\gamma|. \]

Then we have
\[ \int_\Gamma \frac{|\ln|F_t(\gamma)| - \ln|F_t(\tau)||}{(\gamma - t)^2} |d\gamma| \leq |F_t'(t)|. \]

Let $I_2 = \{ t \in D : \omega(|\gamma - \tau|) \leq |F_t(\tau)| \}$. We proceed to estimating the functions $I_2$.

**Estimate for $I_1$.** If $\gamma \in \Gamma_1$, by the mean theorem we have
\[ |\ln|F_t(\gamma)|| - |\ln|F_t(\tau)|| \leq \frac{||F_t(\gamma) - F_t(\tau)||}{\min_{\gamma \in \Gamma_1} (|F_t(\gamma)|, |F_t(\tau)|)} \leq \frac{||f(\gamma) - f(\tau)||}{\min_{\gamma \in \Gamma_1} (|F_t(\gamma)|, |F_t(\tau)|)}, \]

In view of the definition of $\Gamma_1$, we have
\[ |F_t(\gamma)| \geq ||F_t(\gamma) - F_t(\tau)|| \geq |F_t(\tau)| - \frac{1}{2} \omega(|\gamma - \tau|) \geq \frac{1}{2} |F_t(\tau)|. \]

Therefore,
\[ I_1 \leq \frac{1}{1 - |t|} \int \omega(|\gamma - \tau|) P_t(\gamma) |d\gamma|, \]

where $P_t(\gamma)$ is the Poisson kernel.

Employing Lemma 2, we obtain
\[ I_1 \leq \frac{\omega(1 - |t|)}{1 - |t|} \ln \frac{1}{1 - |t|}. \]

By Lemma 1, we finally get
\[ I_1 \leq \frac{\omega(1 - |t|)}{1 - |t|}, \quad t \in D. \]

**Estimate for $I_2$.** Let
\[ K_t(\gamma) = \frac{1}{(\gamma - t)^2}, \quad \gamma \in \Gamma, \quad t \in D. \]

Then we have
\[ I_2 \leq |F_t(\tau)| \max_{\gamma \in \Gamma_2} |K_t(\gamma)| \int_\Gamma |\ln|f_t(\gamma)|| |d\gamma| \leq |F_t(\tau)| \max_{t \in \Gamma_2} |K_t(\gamma)|. \]

In the latter estimate we have employed Lemma 4.

Now, taking into consideration the definition of $\Gamma_2$, we obtain
\[ I_2 \leq |F_t(\tau)| \max_{\gamma \in \Gamma_2} \left\{ \frac{1}{(\gamma - t)^2 (|\gamma - t|^2 + (1 - |t|)^2)^2} \leq |F_t(\tau)| \max_{x \in \Gamma_2} \frac{1}{x^2 : \omega(x) \geq |F_t(\tau)|} \right\}. \]
Let \( x^* \in (0, 2] \) be such that
\[
\omega(x^*) = |F_1(\tau)|.
\] (10)

Then by the latter estimate we get
\[
I_2 \lesssim \frac{|F_1(\tau)|}{(x^*)^2} = C_f \frac{\omega(x^*)}{(x^*)^2}.
\]

The inequality \(|F_1(\tau)| \geq \omega(\sqrt{1-|t|})\) implies \(\sqrt{1-|t|} \leq x^*\). This is why,
\[
\frac{\omega(x^*)}{(x^*)^2} = \frac{\omega(x^*)}{x^* \cdot x^*} \leq \frac{\omega(\sqrt{1-|t|})}{\sqrt{1-|t|}} \frac{1}{\sqrt{1-|t|}} = \frac{\omega(\sqrt{1-|t|})}{1-|t|},
\]
that is,
\[
I_2 \lesssim \frac{\omega(\sqrt{1-|t|})}{1-|t|}.
\]

**Estimate for \(I_3\).** We have
\[
I_3 = |F_1(\tau)| |\ln |F_1(\tau)|| \int_{\Gamma_2} \frac{|d\gamma|}{|\gamma - t|^2} \leq |F_1(\tau)| |\ln |F_1(\tau)|| \int_{\omega(u) \geq |F_1(\tau)|} \frac{du}{u^2 + (1-|t|)^2}
\]
\[
= |F_1(\tau)| |\ln |F_1(\tau)|| \int_{\omega(u) \geq |F_1(\tau)|} \frac{du}{u^2 + (1-|t|)^2} \left( \frac{\pi}{2} - \arg \cot \frac{x^*}{1-|t|} \right),
\]
where \(x^*\) is introduced by identity (10).

Taking into consideration the elementary inequality
\[
0 \leq \frac{\pi}{2} - \arg \cot V \leq \frac{H}{1+V}, \quad V \in [0, +\infty),
\]
by estimate (11) we finally obtain
\[
I_3 \lesssim |F_1(\tau)| |\ln |F_1(\tau)|| \frac{H}{1-|t| + x^*}.
\]

Now, in view of the estimates
\[
\sup_{0 \leq u \leq 2} u |\ln u| \leq e, \quad x^* \geq \sqrt{1-|t|},
\]
by (11) we get
\[
I_3 \lesssim \frac{1}{\sqrt{1-|t|}} \omega(\sqrt{1-|t|}) \frac{\omega(\sqrt{1-|t|})}{1-|t|}.
\]

In the latter inequality we employed the inequality \(\omega(\delta) \geq \omega(1)\) as \(0 < \delta \leq 1\).

2°. **Estimate for \(|F_1(\tau)| |f_1'(t)|\)** As above, we let
\[
K_t(\xi) = \frac{1}{(t-\xi)^2}, \quad \xi \in \Gamma, \quad t \in D.
\]

Then
\[
f_1'(t) = f_1(t) \int_{\Gamma} K_t(\xi) d\mu^t(\xi),
\]
where the measure \(\mu^t\) is supported in the set
\[
E_t = \{ \gamma \in \Gamma : |f(\gamma)| \leq \omega(\sqrt{1-|t|}) \},
\]
at that, \(\mu^t(\Gamma) \leq M\).

Let \(\tau^* \in E_t\) be the point closest to the point \(t\). Then by Lemma 2 we get
\[
|F_1(t)| \lesssim \left[ |F_1(\tau)| + \omega(1-|t|) \ln \frac{1}{1-|t|} \right].
\] (12)
Hence,
\[ |F_i(t)| \lesssim \left( |F_i(\tau) - F_i(\tau^*)| + |F_i(\tau^*)| \omega(1 - |t|) \ln \frac{1}{1 - |t|} \right). \]

Since \( \tau^* \in E_i \), then \( |F_i(\tau^*)| \lesssim \omega(\sqrt{1 - |t|}) \). Therefore, by estimate (12) we obtain
\[ |F_i(t)| \lesssim \left[ \omega(|\tau - \tau^*|) + \omega(1 - |t|) \ln \frac{1}{1 - |t|} + \omega(\sqrt{1 - |t|}) \right]. \]

By Lemma 1 we have
\[ |F_i(t)| \lesssim \left[ \omega(|\tau - \tau^*|) + \omega(\sqrt{1 - |t|}) \right] |J_i(t)| \int_\Gamma |K_i(\xi)| d\mu_i(\xi), \]
that is,
\[ |F_i(t)| |f_i'(t)| \lesssim \left[ |f_i'(t)| \omega(|\tau - \tau^*|) + f_i'(t) \omega(\sqrt{1 - |t|}) \right]. \]

We proceed to estimating the expression in the brackets.

Let
\[ J_1 = |f_i'(t)| \omega(|\tau - \tau^*|) = \omega(|\tau - \tau^*|) |f_i(t)| \int_\Gamma \frac{d\mu_i(\xi)}{|\xi - t|^2}, \]
\[ J_2 = |f_i'(t)| \omega(\sqrt{1 - |t|}). \]

We first estimate \( J_2 \).

We have
\[ J_2 \lesssim \omega(\sqrt{1 - |t|}) \int_\Gamma \frac{d\mu_i(\xi)}{|\xi - t|^2} \exp \left( - \frac{1 - |t|^2}{|\xi - t|^2} d\mu_i(\xi) \right) \]
\[ \lesssim \omega(\sqrt{1 - |t|}) \sup_{u > 0} e^{-u} u \lesssim \omega(\sqrt{1 - |t|}) \quad (13) \]

We proceed to estimating \( J_1 \). If \( \omega(|\tau - \tau^*|) \lesssim \omega(\sqrt{1 - |t|}) \), then \( J_1 \) can be estimated exactly in the same way as \( J_2 \). This is why we assume that \( \omega(\sqrt{1 - |t|}) \lesssim \omega(|\tau - \tau^*|) \). In view of the monotonicity of the function \( \omega \), the latter estimate implies \( \sqrt{1 - |t|} \leq |\tau - \tau^*| \). Therefore, we obtain
\[ J_1 \lesssim \omega(|\tau - \tau^*|) |f_i(t)| \frac{1}{|\tau - \tau^*|^2} \lesssim \frac{\omega(|\tau - \tau^*|)}{|\tau - \tau^*|} \frac{1}{|\tau - \tau^*|} \]
\[ \lesssim \omega(|\tau - \tau^*|) \frac{1}{\sqrt{1 - |t|}} \lesssim \frac{\omega(\sqrt{1 - |t|})}{\sqrt{1 - |t|}} \frac{\omega(\sqrt{1 - |t|})}{1 - |t|}, \quad t \in D. \quad (14) \]

In the latter inequality we have employed the non-increasing of the function \( \frac{\omega(\delta)}{\delta} \) on \((0, 2)\).

Estimates (13), (14) imply the statement of the theorem. \( \square \)

Let us outline the proof of Theorem 2.

Let \( f \) satisfy the assumptions of Theorem 2. We consider the following cut-off function:
\( f(\lambda) = f(\lambda), \lambda \in D, \xi \in S_n \), a point \( \xi \) is fixed, see [11].

It is easy to see that \( f_\xi(\lambda) \) satisfies the assumptions of Theorem 1. In view of Remark 2, we establish the estimate
\[ |f_\xi(\lambda_1) - f_\xi(\lambda_2)| \leq A_\omega \left( \sqrt{|\lambda_1 - \lambda_2|} \right), \quad \lambda_1, \lambda_2 \in D, \]
and \( A \) is independent of \( \xi \in S_n \).
Employing the identity
\[ R(f)(z) = n \int_{S_n} d\sigma(\xi) \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle z, \xi \rangle e^{-i\theta} f(e^{i\theta} \xi) d\theta \frac{1}{(1 - \langle z, \xi \rangle e^{-i\theta})^{n+1}}, \quad z \in B_n, \]
where \( R \) is the radial derivative, see [11], we get the estimate
\[ |R(f)(z)| \lesssim \omega\left(\sqrt{1 - \|z\|}\right), \quad z \in B_n. \]

Arguing as in the proof of Theorem 7.9 in [11], we arrive at the statement of Theorem 2.

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