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ANALYTIC FUNCTIONS WITH SMOOTH ABSOLUTE VALUE OF BOUNDARY DATA

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Abstract. Let f be an analytic function in the unit circle D continuous up to its boundary Γ, $f(z) \neq 0$, $z \in D$. Assume that on Γ, the function f has a modulus of continuity $\omega(|f|, \delta)$. In the paper we establish the estimate $\omega(f, \delta) \leqslant A\omega(|f|, \sqrt{\delta})$, where A is a some non-negative number, and we prove that this estimate is sharp. Moreover, in the paper we establish a multi-dimensional analogue of the mentioned result. In the proof of the main theorem, an essential role is played by a theorem of Hardy-Littlewood type on Hölder classes of the functions analytic in the unit circle.

Keywords: analytic function, modulus of continuity, factorization, outer function.

Mathematics Subject Classification: primary: 30D55, 30D15; secondary: 46E22, 47A15

Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be a bounded unit circle in the complex plane $\mathbb{C}$ and Γ be its boundary. By C_A we denote the set of all functions f analytic in D and continuous in $D \cup \Gamma$. If $f \in C(\Gamma)$, then by the symbol $\omega(f, \delta)$ we denote the modulus of continuity of the function f on Γ, that is,

$$\omega(f, \delta) = \big\{ \sup |f(\gamma) - f(\gamma e^{it})|, \quad \gamma \in \Gamma, \quad |t| \leqslant \delta, \quad t \in \mathbb{R} \big\}.$$

In the paper we consider the following problem: let $f \in C_A$ and at the unit circle the function $|f(e^{i\theta})|$ has the modulus of continuity $\omega(|f|, \delta)$. What is the modulus of continuity of the function f on Γ and hence, on $D \cup \Gamma$?

Such problem in the classes of continuous functions with the modulus of continuity satisfying Bari-Stechkin condition

$$\int\limits_0^\delta \frac{\omega(|f|, \delta)}{t} dt + \delta \int\limits_\delta^\pi \frac{\omega(|f|, \delta)}{t^2} dt = O\big(\omega(|f|, \delta)\big), \quad \delta \to 0, \tag{1}$$

was solved first in work by V.P. Khaving and the author, see [5].

It was established that if $\omega(|f|, \delta)$ satisfies Bari-Stechkin condition (1) and $f(z) \neq 0$, $z \in D$, then

$$\omega(f, \delta) = O\big(\omega(|f|, \sqrt{\delta})\big), \quad \delta \to 0.$$

Moreover, it was shown by simple examples that the obtained estimate was sharp and the condition $f(z) \neq 0$, $z \in D$, is necessary in the known sense. The detailed proof of these statements was exposed in [7]. This work gave rise to rather interesting studies in this direction. First V.P. Khavin, see [6], proposed an interesting approach for obtaining such estimates; this was done by applying the methods of the theory of singular integral operators. Later N.A. Shirokov (see [8], [10]) extended the results of such type for external functions and Hölder

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classes of order $\alpha, \alpha \in (0,+\infty)$ and he obtained the necessary and sufficient condition for $|f(e^{i\theta})|$ ensuring that the function f has a prescribed modulus of continuity on the set $D \cup \Gamma$. In these works there were introduced a new characteristics and in terms of this characteristics, N.A. Shirokov obtained the results of such kind also for the Besov classes of analytic functions in $D \cup \Gamma$. And finally, we mention work [2], where it was established that this phenomenon has a local character, that is, if the modulus of continuity $|f|$ on the circle satisfies the Hölder condition of order α just at one point, then f belongs to the Hölder class of order $\frac{\alpha}{2}$ at this point.

We note that the proof by V.P. Khavni and the proof of the results in works [2], [6], [8], [10] are based on gentle and fine methods of complex and harmonic analysis. In our opinion, the approach applied in works [5] and [7] and is based on classical theorems of Hardy-Littlewood type theorems (see [3], [4]) is more simple. In this work by developing the methods of works [5], [7] we prove such results for the modulus of continuity $\omega(|f|,\delta)$ satisfying the classical Zygmund condition

$$\int_0^\delta \frac{\omega(|f|,t)}{t}dt = O\big(\omega(|f|,\delta)\big), \quad \delta \to 0. \tag{2}$$

The following statement is true.

Theorem 1. *Let f be a function in the class C_A and $f(z) \neq 0$, $z \in D$. If the modulus of the continuity $\omega(|f|,\delta)$ of the function $|f|$ on Γ satisfies Zygmund condition (2), then*

$$\omega(f,\delta) = O\left(\omega(|f|,\sqrt{\delta})\right), \quad \delta \to 0, \tag{3}$$

and this estimate is sharp.

Remark 1. *A simple example, the function*

$$f(z) = (1-z)^{2\alpha} \exp\left(-\frac{1+z}{1-z}\right), \quad z \in D \cup \Gamma, \quad \alpha \in (0,+\infty),$$

where the principal branch of the power function is chosen, shows the sharpness of the statement of Theorem 1.

The analogue of Theorem 1 is true for analytic functions in the unit ball of the space $\mathbb{C}^n$. In order to formulate it, we introduce some notations. Let $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, $\|z\| = \sqrt{|z_1|^2+\ldots+|z_n|^2}$. We define $B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ and $S_n = \{z \in \mathbb{C}^n : \|z\| = 1\}$.

By $H(B_n)$ we denote the set of all analytic functions in B_n. Let $f \in H(B_n)$ and $f(z) = \sum\limits_{k=0}^{+\infty} f_k(z)$ be the expansions of the function f into homogeneous polynomials; by $R(f)$ the radial derivative of the function f [11], that is,

$$R(f)(z) = \sum_{k=1}^{+\infty} k f_k(z), \quad z \in B_n.$$

We introduce also the notation:

$$C_A(B_n) = H(B_n) \cap C(B_n \cup S_n).$$

The following estimate of Theorem 1 is true in the classes $C_A(B_n)$.

Theorem 2. *Let $f \in C_A(B_n)$ and the modulus of continuity $\omega(|f|,\delta)$ of the function $|f|$ on S_n satisfies Zygmund condition (2). Then the modulus of continuity of the function on the set $B_n \cup S_n$ satisfies the estimate*

$$\omega(f,\delta) \leqslant A\omega(|f|,\sqrt{\delta}), \quad 0 \leqslant \delta \leqslant 2.$$

Remark 2. *For Hölder classes, that is, as $\omega(f,t) = t^\alpha, 0 < \alpha \leqslant 1$, $t \in [0,2]$, the analogue of Theorem 2 was established in work [9] by N.A. Shirokov.*

1. Proof of auxiliary statements

Let f and g be real-valued functions with a common domain $E \subset \mathbb{C}$. Then the relation $f \lesssim g$ on E is equivalent to the following: there exists a positive number A such that $f(z) \leqslant Ag(z)$ for each $z \in E$. If $f \lesssim g$ and simultaneously $g \lesssim f$, then $f(z) \approx g(z)$.

In what follows, as a function of modulus of continuity type we call a non-negative non-decreasing function ω on $[0, +\infty)$ such that

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2), \quad \omega(\lambda\delta) \leqslant 2\lambda\omega(\delta), \quad \lambda, \delta \in [0, +\infty).$$

Lemma 1. *Let ω be a function of modulus of continuity type satisfying Zygmund condition (2), then*

$$\omega(\delta) \ln \frac{1}{\delta} \lesssim \omega(\sqrt{\delta}), \quad \delta > 0. \tag{4}$$

Proof. By the definition we have

$$\int_0^{\sqrt{\delta}} \frac{\omega(u)}{u} du \leqslant A\omega(\sqrt{\delta}).$$

It is clear that if $1 \leqslant \delta$, then estimate (4) is obvious and this is why we assume that $0 < \delta < 1$. Then

$$\int_0^{\sqrt{\delta}} \frac{\omega(u)}{u} du = \int_0^{\delta} \frac{\omega(u)}{u} du + \int_\delta^{\sqrt{\delta}} \frac{\omega(u)}{u} du.$$

Hence,

$$\int_0^{\sqrt{\delta}} \frac{\omega(u)}{u} du \geqslant \omega(\delta) \int_\delta^{\sqrt{\delta}} \frac{du}{u} = \frac{\omega(\delta)}{2} \ln \frac{1}{\delta}.$$

It remains to employ Zygmund condition. The proof is complete. □

Lemma 2. *Let $f \in C_A$, $t = |t|\tau$, $t \in D$, $\tau \in \Gamma$. Then the estimate*

$$|f(t)| \lesssim \left(|f(\tau)| + \omega(f, (1 - |t|)) \ln \frac{1}{1 - |t|} \right)$$

is true.

Proof. We have

$$f(t) = \frac{1}{2\pi} \int_\Gamma P_t(\xi) f(\xi) |d\xi|.$$

where $P_t(\xi)$ is the Poisson kernel. This is why

$$|f(t)| \leqslant \frac{1}{2\pi} \int_\Gamma P_t(\xi) |f(\xi) - f(\tau)| |d\xi| + |f(\tau)|.$$

Therefore,

$$|f(t)| \lesssim (|f(\tau)| + J_\omega), \tag{5}$$

where

$$J_\omega := \frac{1}{2\pi} \int_\Gamma P_t(\xi) \omega(f, |\xi - \tau|) d\tau.$$

We proceed to estimating the latter integral. It is clear that

$$J_\omega \lesssim \int_\Gamma \frac{(1-|t|^2)\omega(f,|\xi-\tau|)}{(1-|t|)^2+|\xi-\tau|^2}|d\xi| \lesssim \int_0^\pi \frac{(1-|t|^2)\omega(f,u)}{(1-|t|)^2+u^2}du.$$

We define $\omega(f,\delta)$ on $\mathbb{R}_+ = [0,+\infty)$ as a function of modulus of continuity type (see [1], [4]). Then

$$J_\omega \lesssim \int_0^{\frac{\pi}{1-|t|}} \frac{\omega(f,v(1-|t|))}{1+v^2}dv.$$

Representing this integral as the sum

$$\int_0^1 \frac{\omega(f,v(1-|t|))}{1+v^2}dv + \int_1^{\frac{\pi}{1-|t|}} \frac{\omega(f,v(1-|t|))}{1+v^2}dv,$$

and taking into consideration that $\frac{\omega(f,\delta)}{\delta}$ does not increase (see [1], [4]), we obtain

$$J_\omega \lesssim \omega(f,1-|t|) \int_1^{\frac{\pi}{1-|t|}} \frac{v}{1+v^2}.$$

Hence,

$$J_\omega \lesssim \omega(f,1-|t|)\ln\frac{1}{1-|t|}. \tag{6}$$

By (5), (6) we arrive at the statement of the lemma. □

Lemma 3. *(see [3]). Let λ be a positive non-decreasing function on $(0,1)$ and*

$$\int_0^1 \lambda(u)du < +\infty.$$

Assume that f is an analytic function in D such that

$$\sup_{z\in D}\left\{\frac{|f'(z)|}{\lambda(|z|)}\right\} < +\infty.$$

Then the function f belongs to the class C_A and

$$\omega(f,\delta) \lesssim \int_{1-\delta}^1 \lambda(u)du.$$

Lemma 4. *Let $f \in C_A$, $f(z) \neq 0$, $z \in D, |f(z)| \leqslant 1$, $z \in D$. Then there exists a number $M > 0$ possessing the following properties: for an arbitrary $0 < a < 1$, the function f in D can be represented as*

$$f(z) = \Phi_a(z)\Psi_a(z), z \in D,$$

where Ψ_a is an analytic function in D such that $|\Psi_a|$ is continuously extended to the entire closed circe $D \cup \Gamma$,

$$
\begin{aligned}
&a \leqslant |\Psi_a(t)| \leqslant 1 \quad \textit{for each } t \in D, \\
&\big||\Psi_a(t')| - |\Psi_a(t'')|\big| \leqslant \big||f(t')| - |f(t'')|\big|, \quad t', t'' \in \Gamma, \\
&\int\limits_\Gamma \big|\ln|\Psi_a(t)|\,\big|dt| \leqslant M,
\end{aligned} \tag{7}
$$

$$
\Phi_a(z) = \exp\left(-\int\limits_\Gamma S_z(t)d\mu_a(t)\right), \quad z \in D,
$$

where $S_z(t)$ is the Schwarz kernel for the circle D, μ_a is a non-negative Borel measure Γ, whose total variation does not exceed M.

Proof. Let $H_a(t) = \max(a, |f(t)|)$, $t \in \Gamma$. We have

$$
\Psi_a(z) := \exp\left(\frac{1}{2\pi}\int\limits_\Gamma S_z(t)\ln H_a(t)|dt|\right),
$$

$$
\Phi_a(z) = f(z)\left(\Psi_a(z)\right)^{-1} := \exp\left(-\int\limits_\Gamma S_z(t)d\mu_a(t)\right), \quad z \in D,
$$

where

$$
\mu_a(E) = -\int\limits_{E_a} \ln\left(\frac{|f(t)|}{a}\right)|dt| + \mu(E), \tag{8}
$$

μ is a non-negative measure defining the singular part of the function f, E is an arbitrary Borel set in Γ, $E_a = \{\gamma \in \Gamma : |f(\gamma)| \leqslant a\}$. It is clear that $|\Psi_a|$ on Γ coincides with H_a. Estimate (7) is obtained by the following inequality

$$
\begin{aligned}
\int\limits_\Gamma |\ln|\Psi_a(\xi)|||d\xi| &= \int\limits_{E(|f|\geqslant a)} |\ln|\Psi_a(\xi)|||d\xi| + \int\limits_{E(|f|<a)} |\ln|\Psi_a(\xi)|||d\xi| \\
&= \int\limits_{E(|f|\geqslant a)} |\ln|\Psi_a(\xi)|||d\xi| + \sigma(E_a)\ln\frac{1}{a},
\end{aligned}
$$

where σ is the Lebesgue measure on E_a.

It remains to note that

$$
E_a = E(|f| \leqslant a) = E\left(\ln\frac{1}{|f|} \geqslant \ln\frac{1}{a}\right);
$$

re recall that $\max|f| \leqslant 1$, $0 < a < 1$.

The finiteness of the integral $\int\limits_\Gamma |\ln|f(\xi)|||d\xi|$ implies that

$$
\sup_{A \geqslant 0} A\sigma(\gamma \in \Gamma : |\ln|f(\gamma)|| \geqslant A) < +\infty.
$$

This proves (7).

Now we are going to estimate $\mu_a(E)$. In order to do it, we denote by $V_a(E)$, the first term in the right hand side in (8) and we note that $\mu_a(\Gamma) \leqslant V_a(\Gamma) + \mu(\Gamma)$. This is why

$$
V_a(\Gamma) \leqslant \int\limits_\Gamma |\ln|f||dt + |\ln a|\sigma(\Gamma_a),
$$

where
$$\Gamma_a = \{\gamma \in \Gamma : |\ln|f(\gamma)|| \geqslant |\ln a|\}.$$
Following the lines of the proof of inequality (7), we obtain the last statement of the lemma. The proof is complete. □

Remark 3. *Employing the Jensen inequality, it is easy to observe that if* $\|f\|_{C_A} < 1$, *then*
$$|\mu_a(\Gamma)| \lesssim \ln\frac{1}{|f(0)|}, \qquad \int\limits_{\Gamma} |\ln|\Psi_a|| \, |dt| \lesssim \ln\frac{1}{|f(0)|}.$$

2. Proof of theorems

Proof of Theorem 1. Without loss of generality we assume that $|f(t)| \leqslant 1$, $t \in \Gamma$. Moreover, for the sake of convenience we denote $\omega(\delta) := \omega(|f|, \delta)$, $0 \leqslant \delta \leqslant 2$, and at that,
$$|f(t')| - |f(t'')| \leqslant \frac{1}{2}\omega(|t' - t''|) \quad \text{for all} t', t'' \in \Gamma.$$
Employing Lemma 3, it is sufficient to establish the estimate
$$|f'(t)| \lesssim \frac{\omega(\sqrt{1-|t|})}{1-|t|}, \quad t \in D.$$

Let $t \in D$ be a fixed point in the circle D and in Lemma 4 we choose $a = \omega(\sqrt{1-|t|})$. We introduce the notations
$$F_t(t) = \Psi_{\omega\left(\sqrt{1-|t|}\right)}(t), \quad f_t(t) = \Phi_{\omega\left(\sqrt{1-|t|}\right)}(t).$$

We observe that
$$f'(t) = f_t'(t)F_t'(t) + f_t'(t)F_t(t)$$

1°. *Estimate for* $|f_t(t)||F_t'(t)|$. To estimate this product, let us first prove the inequality
$$|F_t(t)| \lesssim |F_t(\tau)|, \quad t = |t|\tau, \quad \tau \in \Gamma. \tag{9}$$
By Lemma 2,
$$|F_t(t)| \lesssim \left(|F_t(\tau)| + \omega(1-|t|)\ln\frac{1}{1-|t|}\right).$$
This is why to prove inequality (9), it is sufficient to establish the estimate
$$\sup_{t\in D}\{\omega(1-|t|)\ln\frac{1}{1-|t|}\left|F_t^{-1}(\tau)\right|\} < +\infty.$$

Suppose first that
$$\max(|f(\tau)|, \omega(\sqrt{1-|t|})) = |f(\tau)|,$$
then by Lemma 4 we have
$$|F_t(\tau)| \geqslant \omega(\sqrt{1-|t|}).$$
This is why, taking into consideration the estimate
$$\omega(1-|t|)\ln\frac{1}{1-|t|} \lesssim \omega\big(\sqrt{1-|t|}\big),$$
we obtain
$$\frac{\omega(1-|t|)\ln\frac{1}{1-|t|}}{|F_t(\tau)|} \lesssim \frac{\omega(\sqrt{1-|t|})}{\omega(\sqrt{1-|t|})} \lesssim 1.$$

Now we consider the case $|f(\tau)| \leqslant \omega(\sqrt{1-|t|})$. Applying Lemma 1 once again, we get desired estimate (9).

We proceed to estimating the functions $|f_t(t)|$, $\left|F_t'(t)\right|$. Let

$$\Gamma_1 = \{\gamma \in \Gamma : \omega(|\gamma - \tau|) \leqslant |F_t(\tau)|\}, \quad \Gamma_2 = \Gamma \setminus \Gamma_1.$$

Then we have

$$|f_t(t)|\, |F_t'(t)| \leqslant |F_t'(t)|\,.$$

At that,

$$\begin{aligned} |F_t'(t)| &= |F_t(t)| \left| \int\limits_\Gamma |F_t(\gamma)| \frac{2\gamma}{(\gamma - t)^2} |d\gamma| \right| = |F_t(t)| \left| \int\limits_\Gamma \frac{(\ln |F_t(\gamma)| - \ln |F_t(\tau)|)}{(\gamma - t)^2} \right| \\ &\lesssim |F_t(\tau)| \left[\left| \int\limits_{\Gamma_1} \frac{(\ln |F_t(\gamma)| - \ln |F_t(\tau)|)}{(\gamma - t)^2} \right| + \left| \int\limits_{\Gamma_2} \frac{(\ln |F_t(\gamma)| - \ln |F_t(\tau)|)}{(\gamma - t)^2} \right| \right] \\ &\lesssim |F_t(\tau)| \int\limits_{\Gamma_1} \frac{|\ln |F_t(\gamma)| - \ln |F_t(\tau)||}{|\gamma - t|^2} |d\gamma| + |F_t(\tau)| \int\limits_{\Gamma_2} \frac{|\ln |F_t(\gamma)||\, |d\gamma|}{|\gamma - t|^2} \\ &\quad + |F_t(\tau)|\, |\ln |F_t(\tau)|| \int\limits_{\Gamma_2} \frac{|d\gamma|}{|\gamma - t|^2} \stackrel{def}{=} [I_1 + I_2 + I_3]. \end{aligned}$$

Estimate for I_1. If $\gamma \in \Gamma_1$, by the mean theorem we have

$$|\ln |F_t(\gamma)|| - |\ln |F_t(\tau)|| \leqslant \frac{||F_t(\gamma)| - |F_t(\tau)||}{\min\limits_{\gamma \in \Gamma_1} (|F_t(\gamma)|\,, |F_t(\tau)|)} \leqslant \frac{||f(\gamma)| - |f(\tau)||}{\min\limits_{\gamma \in \Gamma_1} (|F_t(\gamma)|\,, |F_t(\tau)|)}.$$

In view of the definition of Γ_1, we have

$$|F_t(\gamma)| \geqslant ||F_t(\tau)| - |F_t(\tau) - F_t(\gamma)|| \geqslant |F_t(\tau)| - \frac{1}{2}\omega(|\gamma - \tau|) \geqslant \frac{1}{2}\, |F_t(\tau)|\,.$$

Therefore,

$$I_1 \lesssim \frac{1}{1 - |t|} \int\limits_\Gamma \omega(|\gamma - \tau|) P_t(\gamma) |d\gamma|,$$

where $P_t(\gamma)$ is the Poisson kernel.

Employing Lemma 2, we obtain

$$I_1 \lesssim \frac{\omega(1 - |t|)}{1 - |t|} \ln \frac{1}{1 - |t|}.$$

By Lemma 1, we finally get

$$I_1 \lesssim \frac{\omega(\sqrt{1 - |t|})}{1 - |t|}, \quad t \in D.$$

Estimate for I_2. Let

$$K_t(\gamma) = \frac{1}{(\gamma - t)^2}, \quad \gamma \in \Gamma, \quad t \in D.$$

Then we have

$$I_2 \leqslant |F_t(\tau)| \max_{\gamma \in \Gamma_2} |K_t(\gamma)| \int\limits_\Gamma |\ln |f_t(\gamma)||\, |d\gamma| \lesssim |F_t(\tau)| \max_{t \in \Gamma_2} |K_t(\gamma)|\,.$$

In the latter estimate we have employed Lemma 4.

Now, taking into consideration the definition of Γ_2, we obtain

$$I_2 \lesssim |F_t(\tau)| \max_{\gamma \in \Gamma_2} \frac{1}{(|\gamma - \tau|^2 + (1 - |t|))^2} \lesssim |F_t(\tau)| \max \left\{ \frac{1}{x^2}, x : \omega(x) \geqslant |F_t(\tau)| \right\}.$$

Let $x^* \in (0, 2]$ be such that

$$\omega(x^*) = |F_t(\tau)| \,. \tag{10}$$

Then by the latter estimate we get

$$I_2 \lesssim \frac{|F_t(\tau)|}{(x^*)^2} = C_f \frac{\omega(x^*)}{(x^*)^2}.$$

The inequality $|F_t(\tau)| \geqslant \omega(\sqrt{1-|t|})$ implies $\sqrt{1-|t|} \leqslant x^*$. This is why,

$$\frac{\omega(x^*)}{(x^*)^2} = \frac{\omega(x^*)}{x^* \cdot x^*} \leqslant \frac{\omega(\sqrt{1-|t|})}{\sqrt{1-|t|}} \frac{1}{\sqrt{1-|t|}} = \frac{\omega(\sqrt{1-|t|})}{1-|t|},$$

that is,

$$I_2 \lesssim \frac{\omega(\sqrt{1-|t|})}{(1-|t|)}.$$

Estimate for I_3. We have

$$\begin{aligned} I_3 &= |F_t(\tau)| \, |\ln |F_t(\tau)|| \int\limits_{\Gamma_2} \frac{|d\gamma|}{|\gamma - t|^2} \lesssim |F_t(\tau)| \, |\ln |F_t(\tau)|| \int\limits_{\omega(u) \geqslant |F_t(\tau)|} \frac{du}{u^2 + (1-|t|)^2} \\ &= |F_t(\tau)| \, |\ln |F_t(\tau)|| \lesssim \frac{|F_t(\tau)| \, |\ln |F_t(\tau)||}{(1-|t|)} \left(\frac{\pi}{2} - \operatorname{arg\,cot} \frac{x^*}{1-|t|} \right), \end{aligned} \tag{11}$$

where x^* is introduced by identity (10).

Taking into consideration the elementary inequality

$$0 \leqslant \frac{\pi}{2} - \operatorname{arg\,cot} V \leqslant \frac{H}{1+V}, \quad V \in [0, +\infty),$$

by estimate (11) we finally obtain

$$I_3 \lesssim |F_t(\tau)| \, |\ln |F_t(\tau)|| \, \frac{H}{1-|t|+x^*}.$$

Now, in view of the estimates

$$\sup_{0 \leqslant u \leqslant 2} u |\ln u| \leqslant e, \quad x^* \geqslant \sqrt{1-|t|},$$

by (11) we get

$$I_3 \lesssim \frac{1}{\sqrt{1-|t|}} \frac{\omega(\sqrt{1-|t|})}{\sqrt{1-|t|}} = C_f \frac{\omega(\sqrt{1-|t|})}{1-|t|}.$$

In the latter inequality we employed the inequality $\frac{\omega(\delta)}{\delta} \geqslant \omega(1)$ as $0 < \delta \leqslant 1$.

2°. *Estimate for* $|F_t(\tau)| \, |f_t'(t)|$ As above, we let

$$K_t(\xi) = \frac{1}{(t-\xi)^2}, \quad \xi \in \Gamma, \quad t \in D.$$

Then

$$f_t'(t) = f_t(t) \int\limits_{\Gamma} K_t(\xi) d\mu^t(\xi),$$

where the measure μ^t is supported in the set

$$E_t = \left\{ \gamma \in \Gamma : |f(\gamma)| \leqslant \omega(\sqrt{1-|t|}) \right\},$$

at that, $\mu^t(\Gamma) \leqslant M$.

Let$\tau^* \in E_t$ be the point closest to the point t. Then by Lemma 2 we get

$$|F_t(t)| \lesssim \left[|F_t(\tau)| + \omega(1-|t|) \ln \frac{1}{1-|t|} \right]. \tag{12}$$

Hence,

$$|F_t(t)| \lesssim \left(|F_t(\tau) - F_t(\tau^*)| + |F_t(\tau^*)|\, \omega(1-|t|) \ln \frac{1}{1-|t|} \right).$$

Since $\tau^* \in E_t$, then $|F_t(\tau^*)| \leqslant \omega(\sqrt{1-|t|})$. Therefore, by estimate (12) we obtain

$$|F_t(t)| \lesssim \left[\omega(|\tau - \tau^*|) + \omega(1-|t|) \ln \frac{1}{1-|t|} + \omega(\sqrt{1-|t|}) \right].$$

By Lemma 1 we have

$$|F_t(t)|\, |f_t'(t)| \lesssim \left[\omega(|\tau-\tau^*|) + \omega(\sqrt{1-|t|}) \right] |f_t(t)| \int\limits_{\Gamma} |K_t(\xi)|\, d\mu^t(\xi),$$

that is,

$$|F_t(t)|\, |f_t'(t)| \lesssim \left[|f_t'(t)|\, \omega(|\tau-\tau^*|)\, |+ f_t'(t)|\, \omega(\sqrt{1-|t|}) \right].$$

We proceed to estimating the expression in the brackets.

Let

$$J_1 = |f_t'(t)|\, \omega(|\tau-\tau^*|) = \omega(|\tau-\tau^*|)\, |f_t(t)| \int\limits_{\Gamma} \frac{d\mu^t(\xi)}{|\xi - t|^2},$$

$$J_2 = |f_t'(t)|\, \omega(\sqrt{1-|t|}).$$

We first estimate J_2.

We have

$$\begin{aligned} J_2 \lesssim& \omega(\sqrt{1-|t|}) \int\limits_{\Gamma} \frac{d\mu^t(\xi)}{|\xi-t|^2} \exp\left(-\frac{1-|t|^2}{|\xi-t|^2} d\mu^t(\xi) \right) \\ \lesssim& \frac{\omega(\sqrt{1-|t|})}{1-|t|} \sup_{u>0} e^{-u} u \lesssim \frac{\omega(\sqrt{1-|t|})}{1-|t|}. \end{aligned} \tag{13}$$

We proceed to estimating J_1. If $\omega(|\tau-\tau^*|) \lesssim \omega(\sqrt{1-|t|})$, then J_1 can be estimated exactly in the same way as J_2. This is why we assume that $\omega(\sqrt{1-|t|}) \leqslant \omega(|\tau-\tau^*|)$. In view of the monotonicity of the function ω, the latter estimate implies $\sqrt{1-|t|}) \leqslant |\tau-\tau^*|$. Therefore, we obtain

$$\begin{aligned} J_1 \lesssim& \omega(|\tau-\tau^*|) |f_t(t)| \frac{1}{|t-\tau^*|^2} \lesssim \frac{\omega(|\tau-\tau^*|)}{|t-\tau^*|} \frac{1}{|t-\tau^*|} \\ \lesssim& \frac{\omega(|\tau-\tau^*|)}{|\tau-\tau^*|} \frac{1}{\sqrt{1-|t|}} \lesssim \frac{\omega(\sqrt{1-|t|})}{(\sqrt{1-|t|})(\sqrt{1-|t|})} = \frac{\omega(\sqrt{1-|t|})}{1-|t|}, \quad t \in D. \end{aligned} \tag{14}$$

In the latter inequality we have employed the non-increasing of the function $\frac{\omega(\delta)}{\delta}$ on $(0,2)$.

Estimates (13), (14) imply the statement of the theorem. □

Let us outline the proof of Theorem 2.

Let f satisfy the assumptions of Theorem 2. We consider the following cut-off function: $f(\lambda) = f(\lambda\xi)$, $\lambda \in D$, $\xi \in S_n$, a point ξ is fixed, see [11].

It is easy to see that $f_\xi(\lambda)$ satisfies the assumptions of Theorem 1. In view of Remark 2, we establish the estimate

$$|f_\xi(\lambda_1) - f_\xi(\lambda_2)| \leqslant A\omega\left(\sqrt{|\lambda_1 - \lambda_2|} \right), \quad \lambda_1, \lambda_2 \in D,$$

and A is independent of $\xi \in S_n$.

Employing the identity

$$R(f)(z) = n \int_{S_n} d\sigma(\xi) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\langle z, \xi \rangle e^{-i\theta} f(e^{i\theta}\xi) d\theta}{(1 - \langle z, \xi \rangle e^{-i\theta})^{n+1}}, \quad z \in B_n,$$

where R is the radial derivative, see [11], we get the estimate

$$|R(f)(z)| \lesssim \frac{\omega(\sqrt{1 - \|z\|})}{1 - \|z\|}, \quad z \in B_n.$$

Arguing as in the proof of Theorem 7.9 in [11], we arrive at the statement of Theorem 2.

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