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Multiple interpolation and principal parts of a Laurent series for meromorphic functions in the unit disk with power growth of the Nevanlinna characteristic

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Response to Reviewers:	Dear reviewer and editor, your comments were taken into account. We've reduced the calculations in the proof of theorems and proofs removed for obvious lemmas.

Multiple interpolation and principal parts of a Laurent series for meromorphic functions in the unit disk with power growth of the Nevanlinna characteristic

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Abstract. In this paper we solve the multiple interpolation problem in the class of analytic functions in the unit disk with power growth of the Nevanlinna characteristic under the condition that interpolation nodes are contained in a finite union of Stolz angles and describe the principal parts of a Laurent series of meromorphic functions with the same restrictions on the Nevanlinna characteristic.

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1. Introduction

Let $\mathbb{C}$ be the complex plane, D be the unit disk on $\mathbb{C}$, $H(D)$ be the set of all functions, holomorphic in D, $M(D)$ be the set of all functions, meromorphic in D. For any $\alpha > 0$ we define the class S_α^∞ as:

$$S_\alpha^\infty := \left\{ f \in M(D) : T(r, f) \leq \frac{C_f}{(1-r)^\alpha} \right\},$$

where $C_f > 0$ is a positive constant, values of which depend on the function f, $r \in [0, 1)$, $T(r, f)$ is the Nevanlinna characteristic of the function f (see [7]):

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(re^{i\varphi})| d\varphi + N(r, f),$$

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$N(r,f) = \int\limits_0^r \frac{n(t,\infty)-n(0,\infty)}{t}dt$, $0 < r < 1$, $n(t,\infty)$ is the number of poles in the disk $|z| < t$, $0 < t < 1$; $n(0,\infty)$ is the multiplicity of the pole at the point $z = 0$, $a^+ = \max(a, 0)$, $a \in \mathbb{R}$. We also denote

$$S^\infty_{\alpha,a} := S^\infty_\alpha \bigcap H(D).$$

It worth mentioned that classes S^∞_α first was emerged in the famous works of Rolf Nevanlinna (see [7]). He tried to disseminate the results of J. Hadamard and E. Borel (see [4]) to the case of meromorphic functions in a disk. He proved that if $\{a_k\}$, $\{b_k\}$ are the set of zeros and poles of certain function from the class S^∞_α, then the series converges:

$$\sum_{k=1}^{+\infty}(1-|a_k|)^{\alpha+2+\varepsilon} < +\infty, \sum_{k=1}^{+\infty}(1-|b_k|)^{\alpha+2+\varepsilon} < +\infty,$$

for an arbitrary $\varepsilon > 0$. Attempts to obtain a complete description of these sets have also been made by famous Japanese mathematician M. Tsuji (see [16, 17]). This problem was finally solved by F. A. Shamoyan and E. N. Shubabko (see [12]). Namely the following result was provided:
In order to $a = \{a_k\}_1^\infty, b = \{b_k\}_1^\infty \subset D$ are the set of zero and poles certain function $f \in S^\infty_\alpha$, it is necessary and sufficient, that

$$n(r,a) = card\{a_k : |a_k| < r\} \le \frac{c_1}{(1-r)^{\alpha+1}},$$

$$n(r,b) = card\{b_k : |b_k| < r\} \le \frac{c_2}{(1-r)^{\alpha+1}},$$

$0 < r < 1,\ c_1, c_2 > 0.$
They also constructed factorization theory of the class S^∞_α.

It is well known that if $f \in S^\infty_{\alpha,a}$, then

$$M(r,f) = \max_{|z|\le r}|f(z)| \le \exp\left\{\frac{c_f}{(1-r)^{\alpha+1}}\right\} \tag{1.1}$$

for all $\alpha > 0$, $c_f > 0$ (see [7]).

State the problem of multiple interpolation for the class $S^\infty_{\alpha,a}$. Let $\{\alpha_k\}_1^\infty$ and $\{\gamma_k\}_1^\infty$ be the arbitrary sequences of complex numbers from D; put p_j be the multiplicity of the number α_j in the $\{\alpha_k\}_1^\infty$, $s_j \ge 1$ be the multiplicity of the number α_j on the interval $\{\alpha_k\}_1^j$. Obviously, $1 \le s_j \le p_j \le +\infty$. We need to find the criteria for $\{\alpha_k\}_1^\infty$ and $\{\gamma_k\}_1^\infty$, providing the existence of a function $f \in S^\infty_\alpha$, such that

$$f^{(s_k-1)}(\alpha_k) = \gamma_k,\ k = 1, 2, ...$$

Let us note that interpolation theory has become intensively developed since Carleson's fundamental work (1958) on interpolation in the class of bounded analytic functions (see [1]). The term of *free interpolation* was first introduced in [18] by Vinogradov and Havin (1974). The interpolation problem in subclasses of the bounded type functions N was investigated there. The same problem in the Nevanlinna and Smirnov classes was solved in [6, 5] by

Naftalevic (1956) and Hartmann at al. (2004). These questions in the Hardy and Bergman spaces was studied in works of Shapiro and Shields (1961), Seip (2004) (see [14, 8]). Multiple interpolation problem in Hardy classes H^p was solved in the works of M. Djrbashian and Ayrapetian (see [3]).

The paper is organized as follows: in the next section we present the formulation of main result of the article and prove some auxiliary results, in the second section we present the proof of main result.

2. Formulation of main result and proof of auxiliary results

To formulate and proof the results of the work we introduce some more notations and definitions.

For any $\beta > -1$ we denote $\pi_\beta(z, \alpha_k)$ as M. M. Djrbashian's infinite product with zeros at points of the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ (see [2]):

$$\pi_\beta(z, \alpha_k) = \prod_{k=1}^{+\infty} \left(1 - \frac{z}{\alpha_k}\right) \exp(-U_\beta(z, \alpha_k)), \tag{2.1}$$

where

$$U_\beta(z, \alpha_k) = \frac{2(\beta+1)}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2)^\beta \ln|1 - \frac{\rho e^{i\theta}}{\alpha_k}|}{(1 - z\rho e^{-i\theta})^{\beta+2}} d\theta \rho d\rho.$$

We denote $\pi_{\beta,n}(z, \alpha_k)$ as infinite product $\pi_\beta(z, \alpha_k)$ without n-th factor.

As stated in [2], the infinite product $\pi_\beta(z, \alpha_k)$ is absolutely and uniformly convergent in the unit disk D if and only if the series converges:

$$\sum_{k=1}^{+\infty} (1 - |\alpha_k|)^{\beta+2} < +\infty.$$

If $\beta + 1 = p \in \mathbb{Z}_+$ then product (2.1) takes a form (see [2]):

$$\pi_p(z, \alpha_k) = \prod_{k=1}^{+\infty} \frac{\overline{\alpha}_k(\alpha_k - z)}{1 - \overline{\alpha}_k z} \exp \sum_{j=1}^{p+1} \frac{1}{j} \left(\frac{1 - |\alpha_k|^2}{1 - \overline{\alpha}_k z} \right)^j.$$

Definition 2.1. The angle with vertex $e^{i\theta}$, contained in D, having opening $\pi\delta$, $0 < \delta < 1$, and bisector $re^{i\theta}$, $0 \le r < 1$, is said to be the Stolz angle $\Gamma_\delta(\theta)$.

The sequence $\{z_j\}_{j=1}^{+\infty} \subset D$ under following conditions

$$n(r) = card\{z_k : |z_k| < r\} \le \frac{c}{(1-r)^{\alpha+1}}, \tag{2.2}$$

$$|\pi_{p,n}(z_n, z_j)| \ge \exp \frac{-c_0}{(1 - |z_n|)^{\alpha+1}}, c_0 > 0, \tag{2.3}$$

$$\sup_{k\ge1}\{p_k\} = n,$$

we associate with the class $\widetilde{\Delta}$.

The main result of this article is the proof of the following theorems:

Theorem 2.2. *Let* $\{\alpha_k\} \subset \bigcup\limits_{s=1}^{n} \Gamma_\delta(\theta_s)$ *for certain* δ, $0 < \delta < \frac{1}{\alpha+1}$.
If $\{\alpha_k\}_1^\infty \in \widetilde{\Delta}$, *then for any sequence* $\{\gamma_k\}_1^\infty$ *under the condition*

$$|\gamma_k| \le \exp \frac{c_0}{(1-|\alpha_k|)^{\alpha+1}}, k = 1, 2, \dots, c_0 > 0, \tag{2.4}$$

it is possible to construct the function $f \in S^\infty_{\alpha,a}$ *that solve the multiple interpolation problem*

$$f^{(s_k-1)}(\alpha_k) = \gamma_k, k = 1, 2, \dots, \tag{2.5}$$

for all $s_k \ge 1$.

Conversely, if the multiple interpolation problem (2.5) *is solvable for all* $s_k \ge 1$ *and* $\{\gamma_k\}_1^\infty$, *satisfying the condition* (2.4), *then interpolation nodes* $\{\alpha_k\}_1^\infty$ *belong to the class* $\widetilde{\Delta}$.

Theorem 2.3. *Let* $\{z_k\} \subset \bigcup\limits_{s=1}^{n} \Gamma_\delta(\theta_s)$ *for certain* δ, $0 < \delta < \frac{1}{\alpha+1}$. *If* $\{z_k\}_1^\infty \in \widetilde{\Delta}$, *then a necessary and sufficient condition for the existence of function* $F \in S^\infty_\alpha$ *with the following principal parts*

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z-z_k)^n} + \frac{a_{k,n-1}}{(z-z_k)^{n-1}} \dots + \frac{a_{k,1}}{(z-z_k)}, k = 1, 2, \dots \tag{2.6}$$

is that

$$|a_{k,i}| \le \exp \frac{c}{(1-|z_k|)^{\alpha+1}}, i = \overline{1, n}, \tag{2.7}$$

where $c \ne c(i)$.

The proofs of the theorems are based on the following statements.

Theorem A.(see [13]) *Let* $\{\alpha_k\}_{k=1}^{+\infty}$ *be the arbitrary sequence of complex numbers from* D, *which is contained in a finite union of Stolz angles, i.e.*

$$\{\alpha_k\} \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s),$$

with certain $0 < \delta < \frac{1}{\alpha+1}$.
The following statements are equivalent:

1. $\{\alpha_k\}_{k=1}^{+\infty}$ *is interpolating sequence in* S^∞_α, $\alpha > 0$,
2. $$n(r) = card\{\alpha_k : |\alpha_k| < r\} \le \frac{c}{(1-r)^{\alpha+1}},$$
   for some $c > 0$;
3. $$|\pi'_\beta(\alpha_n, \alpha_k)| \ge \exp \frac{-M}{(1-|\alpha_n|)^{\alpha+1}},$$
   for some $M > 0$ *and all* $\beta > \alpha - 1$.

Here and in the sequel, unless otherwise noted, we denote by c, $c_1, \dots$, $c_n(\alpha, \beta, \dots)$ some arbitrary positive constants depending on α, $\beta, \dots$, whose specific values are immaterial.

Lemma 2.4. (see [9]) *If members of the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ are contained in a finite union of Stolz angles, i.e.* $\{\alpha_k\} \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s)$ *for certain* δ, $0 < \delta < \frac{1}{\alpha+1}$, *then for any function* $g_{\alpha+1}(z) = \prod_{s=1}^{n} \exp \frac{C}{(1-ze^{-i\theta_s})^{\alpha+1}}$, $z \in D$, *the following estimate is valid*

$$|g_{\alpha+1}(\alpha_k)| \geq c_1 \exp \frac{c_2}{(1-|\alpha_k|)^{\alpha+1}}, k = 1, 2, ..., \tag{2.8}$$

where c_1, c_2 *are some positive constants.*

We denote

$$K_\rho(\alpha_n) := \left\{ z \in D : |z - \alpha_n| < \exp \frac{-\rho}{(1-|\alpha_n|)^{p+1}} \right\}, \ |\alpha_n| < 1, \ \rho > 0.$$

Lemma 2.5. (see [9]) *Suppose* $\{\alpha_k\} \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s)$ *with certain* δ, $0 < \delta < \frac{1}{\alpha+1}$; *then the following estimate is valid:*

$$\max_{t \in K_\rho(\alpha_k)} |g_{\alpha+1}^{-1}(t)| \leq \widetilde{c} \cdot |g_{\alpha+1}^{-1}(\alpha_k)|, \ k = 1, 2, \dots$$

Lemma 2.6. *For any* $z \in K_\rho(\alpha_n)$ *the following estimate is valid:*

$$\frac{1}{2}|m_j(\alpha_n)| \leq |m_j(z)| \leq \frac{3}{2}|m_j(\alpha_n)|,$$

where

$$m_j(z) = \left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z} \right).$$

The proof of Lemma 2.6 is trivial.

Lemma 2.7. *If the sequence* $\{\alpha_j\} \subset D$ *satisfy the condition* (2.2), *then for any* $z \in D$ *and* $p > \alpha$ *the following estimate is valid:*

$$\sum_{j=1}^{+\infty} \left(\frac{1-|\alpha_j|^2}{|1-\alpha_j z|} \right)^{p+1} \leq \frac{c}{(1-r)^{\alpha+1}}.$$

Really, if $p > \alpha$, then from (2.2) it follows that

$$\sum_{k=1}^{+\infty} (1-|\alpha_k|)^{p+1} < +\infty. \tag{2.9}$$

The required estimate is established in the same way as in the work [11].

Lemma 2.8. *Suppose* $\{\alpha_k\} \subset \widetilde{\Delta}$; *then there exists* $\rho > 0$, $\rho = \rho(c_0)$, *such that for any* $z \in K_\rho(\alpha_n)$, $n = 1, 2, ...$, *the following estimate is valid*

$$|\pi_{p,n}(z, \alpha_k)| \geq \exp \frac{-C}{(1-|\alpha_n|)^{\alpha+1}}.$$

Proof. Fix a number $n \in \mathbb{N}$. Estimate the product $\pi_{p,n}(z,\alpha)$ in the disk $K_\rho(\alpha_n)$. We have

$$\ln \pi_{p,n}(z,\alpha) = \sum_{j=1, j\neq n}^{+\infty} \ln A_j(z,\alpha_j) =$$
$$= \sum_{\substack{j=1\\ j\neq n}}^{+\infty} \left[\ln\left(1 - \frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right) + \sum_{s=1}^{p} \frac{1}{s}\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right)^s \right],$$

where a principal branch of the logarithmic function is taken.

We split the sum $\Sigma = \sum\limits_{\substack{j=1\\ j\neq n}}^{+\infty} \ln A_j(z,\alpha_j)$ into two parts:

$$\Sigma = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{|m_j(z)|\leq \frac{1}{2}} \ln A_j(z,\alpha_j),$$
$$\Sigma_2 = \sum_{|m_j(z)|> \frac{1}{2}} \ln A_j(z,\alpha_j).$$

It is clear that

$$\Sigma_1 = \sum_{\substack{j=1\\ j\neq n}}^{+\infty} \sum_{s=p+1}^{+\infty} \frac{1}{s}(m_j(z))^s = \sum_{\substack{j=1\\ j\neq n}}^{+\infty} (m_j(z))^{p+1} \cdot \left(\sum_{s=p+1}^{+\infty} \frac{1}{s}(m_j(z))^{s-p-1} \right).$$

Therefore,we have

$$|\Sigma_1| = \left| \sum_{|m_j(z)|\leq\frac{1}{2}} \ln A_j(z,\alpha_j) \right| \leq$$
$$\leq \sum_{|m_j(z)|\leq\frac{1}{2}} |m_j(z)|^{p+1} \sum_{s=p+1}^{+\infty} \frac{1}{s}|m_j(z)|^{s-p-1} \leq$$
$$\leq \sum_{|m_j(z)|\leq\frac{1}{2}} |m_j(z)|^{p+1} \sum_{k=0}^{+\infty} \frac{1}{2^k} \leq$$
$$\leq 2 \sum_{|m_j(z)|\leq\frac{1}{2}} \frac{(1-|\alpha_j|^2)^{p+1}}{|1-\alpha_j z|^{p+1}}.$$

Using Lemma 2.7, we get

$$|\Sigma_1| \leq \frac{c_1}{(1-r)^{\alpha+1}}.$$

Now we estimate the sum Σ_2. First we remark that if $|m_j(z)| > \frac{1}{2}$, then $|m_j(\alpha_n)| > \frac{1}{2}$ in the disk $K_\rho(\alpha_n)$ for sufficiently large ρ. We obtain a lower bound for $\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right|$:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| = \left[\ln\frac{|z-\alpha_j|}{|\alpha_n-\alpha_j|} + \ln\left|\frac{1-\overline{\alpha}_j\alpha_n}{1-\overline{\alpha}_j z}\right|\right] + \\ + \sum_{s=1}^{p}\frac{1}{s}\Re\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right)^s - \sum_{s=1}^{p}\frac{1}{s}\Re\left(\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j\alpha_n}\right)^s.$$

Choose ρ sufficiently large to satisfy the following inequality

$$\frac{1}{2} \le \left|\frac{z-\alpha_j}{\alpha_n-\alpha_j}\right| = \left|1+\frac{z-\alpha_n}{\alpha_n-\alpha_j}\right| \le \frac{3}{2}.$$

Taking into account Lemma 2.6, we obtain:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -\ln 2 + |m_j(z)|^{p+1}\times \\ \times\left[\sum_{s=1}^{p}\frac{1}{s}\Re(m_j(z))^s\cdot\frac{1}{|m_j(z)|^{p+1}} - \sum_{s=1}^{p}\frac{1}{s}\Re(m_j(\alpha_n))^s\cdot\frac{1}{|m_j(z)|^{p+1}}\right].$$

Since $|\Re w| \le |w|$, $w\in\mathbb{C}$, we have:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -\ln 2 - |m_j(z)|^{p+1}\times \\ \times\left[\sum_{s=1}^{p}\frac{1}{s}\frac{1}{|m_j(z)|^{p+1-s}} + \sum_{s=1}^{p}\frac{1}{s}\frac{1}{|m_j(\alpha_n)|^{p+1-s}}\right]$$

Since $|m_j(z)| > \frac{1}{2}$ and $|m_j(\alpha_n)| > \frac{1}{2}$ in the disk $K_\rho(\alpha_n)$, then the last inequality is equivalent to

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -\ln 2 - |m_j(z)|^{p+1}\times\sum_{s=1}^{p}\frac{1}{s}2\cdot 2^{p+1-s} \ge \\ \ge -\ln 2 - 2^{p+2}|m_j(z)|^{p+1}\times\sum_{s=1}^{p}\frac{1}{s\cdot 2^s},$$

whence we conclude:

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -|m_j(z)|^{p+1}\cdot\left(\frac{\ln 2}{|m_j(z)|^{p+1}} + 2^{p+2} - 4\right) \ge \\ \ge -|m_j(z)|^{p+1}\cdot\left(2^{p+1}\ln 2 + 2^{p+2} - 4\right),$$

i.e.

$$\ln\left|\frac{A_j(z,\alpha_j)}{A_j(\alpha_n,\alpha_j)}\right| \ge -c_2(p)|m_j(z)|^{p+1}.$$

Now by (2.3), we obtain

$$\ln|A_j(z,\alpha_j)| \ge -c_2(p)\left|\frac{1-|\alpha_j|^2}{1-\overline{\alpha}_j z}\right|^{p+1} - \frac{c_0}{(1-|\alpha_n|)^{p+1}}.$$

Integrating the estimates for the sum Σ_1 and Σ_2, we obtain:

$$\sum_{\substack{j=1\\ j\neq n}}^{+\infty} \ln|A_j(z,\alpha_j)| = \sum_{|m_j(z)|\leq\frac{1}{2}} \ln|A_j(z,\alpha_j)| + \sum_{|m_j(z)|>\frac{1}{2}} \ln|A_j(z,\alpha_j)| \geq$$

$$\geq -\frac{c_1}{(1-|\alpha_n|)^{\alpha+1}} - \frac{c_2(p)}{(1-|\alpha_n|)^{\alpha+1}} - \frac{c_0}{(1-|\alpha_n|)^{\alpha+1}}.$$

Finally, we conclude:

$$\ln|\pi_{p,n}(z,\alpha_n)| = \sum_{\substack{j=1\\ j\neq n}}^{+\infty} |\ln A_j(z,\alpha_j)| \geq \frac{-C}{(1-|\alpha_n|)^{\alpha+1}}.$$

□

To formulate and proof further results, we introduce some more notations.

First we remark that the function $\pi_{p,k}(z)\cdot\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1}$ is analytic in D and is non-zero in the certain neighborhood of the point $z=\alpha_k$ for all $p>\alpha$. For any $k\in\mathbb{N}$ we consider the function

$$\tau_k(z) = \left\{\pi_{p,k}(z)\cdot\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p_k+1}\cdot g_{\alpha+1}(z)\right\}^{-1},$$

where $g_{\alpha+1}(z) = \prod_{s=1}^{n} \exp\frac{C}{(1-ze^{-i\theta_s})^{\alpha+1}}$ and C is sufficient large number. It can be argued that in the sufficient small ε-neighborhood of the point α_k the following expansion is valid:

$$\tau_k(z) = \sum_{\nu=0}^{\infty} a_\nu(\alpha_k)(z-\alpha_k)^\nu, |z-\alpha_k|<\varepsilon,$$

where $a_\nu(\alpha_k) = \frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left[\left\{\pi_{p,k}(z)\cdot\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha_k}z}\right)^{p+p_k+1}\cdot g_{\alpha+1}(z)\right\}^{-1}\right]_{z=\alpha_k}$.

Lemma 2.9. *If $\{\alpha_k\}_1^\infty\in\widetilde{\Delta}$, then for the coefficients of expansion $a_\nu(\alpha_k)$ the following estimates are valid*

$$|a_\nu(\alpha_k)|\leq a(\nu),\ 0\leq\nu\leq p_k,\ k=1,2,\ldots,$$

where $a(\nu)$ depends on the ν.

The proof of the Lemma 2.9 is conducted in a standard way with using Lemma 2.8 (see [10]).

Consider the polynomials

$$q_k(z) = \sum_{\nu=0}^{p_k-s_k} a_\nu(\alpha_k)(z-\alpha_k)^\nu, k=1,2,\ldots$$

Now we define the system of analytic functions in D:

$$\widetilde{\Omega}_k(z) = \frac{(z-\alpha_k)^{s_k-1} q_k(z)}{(s_k-1)!\tau_k(z)}. \tag{2.10}$$

It obvious that

$$\widetilde{\Omega}_k(z) = g_{\alpha+1}(z)\frac{\pi_{p,k}(z)}{(s_k-1)!}\cdot\left(\frac{1-|\alpha_k|^2}{1-\overline{\alpha}_k z}\right)^{p+p_k+1}\cdot\sum_{\nu=0}^{p_k-s_k} a_\nu(\alpha_k)(z-\alpha_k)^{\nu+s_k-1},$$

$k=1,2,\ldots$, where $p>\alpha$.

We note that the method of constructing such system of functions was first proposed by M. Djrbashian in [3].

Lemma 2.10. *Functions of the system* (2.10) *have the following interpolating properties:*

$$\widetilde{\Omega}_k^{(r)}(\alpha_k) = \begin{cases} 1, & r = s_k - 1; \\ 0, & r \neq s_k - 1,\ 0 \leq r \leq p_k - 1. \end{cases}$$

3. Proof of main result

Proof. of Theorem 2.2.

Let interpolation nodes satisfy the following condition: $\{\alpha_k\}\subset\bigcup\limits_{s=1}^{n}\Gamma_\delta(\theta_s)$ with certain $0<\delta<\frac{1}{\alpha+1}$ and $\{\alpha_k\}_1^\infty\in\widetilde{\Delta}$. For any $\{\gamma_k\}_1^\infty$ under condition (2.4) we construct the interpolation function $f(z)$ as follows:

$$f(z) = g_{\alpha+1}(z)\times\sum_{k=1}^{+\infty}\widetilde{\Omega}_k(z)\frac{\gamma_k}{g_{\alpha+1}(\alpha_k)} = g_{\alpha+1}(z)\cdot\varphi(z),\ z\in D,$$

where

$$g_{\alpha+1}(z) = \prod_{s=1}^{n}\exp\frac{C}{(1-ze^{-i\theta_s})^{\alpha+1}}, z\in D. \tag{3.1}$$

Using Lemma 2.9, we get: $f^{(s_k-1)}(\alpha_k)=\gamma_k, k=1,2,\ldots$

Now we prove that the function $f(z)$ belongs to the class $S^\infty_{\alpha,a}$. We obtain an upper estimate on $\varphi(z)$. Since the interpolation nodes satisfy the following condition: $\{\alpha_k\}\subset\bigcup\limits_{s=1}^{n}\Gamma_\delta(\theta_s)$, and the elements of the sequence $\{\gamma_k\}_1^{+\infty}$ satisfy the condition (2.4), then splitting the sum $\varphi(z)$ on n parts and applying to each of them Lemma 2.4, we obtain:

$$|\varphi(z)| = \sum_{s=1}^{n}\sum_{\alpha_k\in\Gamma_\delta(\theta_s)}\left|\frac{\gamma_k}{g_{\alpha+1}(\alpha_k)}\right|\cdot|\widetilde{\Omega}_k(z)| \leq$$

$$\leq c_0\sum_{s=1}^{n}\sum_{\alpha_k\in\Gamma_\delta(\theta_s)}\exp\frac{\delta-C}{(1-|\alpha_k|)^{\alpha+1}}\cdot|\widetilde{\Omega}_k(z)|.$$

Choosing a positive constant C such that $\delta - C < 0$, we obtain the following estimate:

$$\exp \frac{\delta - C}{(1-|\alpha_k|)^{\alpha+1}} \leq 1,$$

for all $k = 1, 2, ...$

So, we have:

$$|\varphi(z)| \leq c_0 \sum_{k=1}^{+\infty} |\widetilde{\Omega}_k(z)|.$$

We obtain an upper estimate of the function $\widetilde{\Omega}_k(z)$ for all $k = 1, 2, \ldots$ Recall that

$$|\widetilde{\Omega}_k(z)| = |g_{\alpha+1}(z)| \frac{|\pi_{p,k}(z)|}{(s_k-1)!} \cdot \left(\frac{1-|\alpha_k|^2}{|1-\overline{\alpha}_k z|} \right)^{p+p_k+1} \times$$
$$\times \left| \sum_{\nu=0}^{p_k-s_k} a_\nu(\alpha_k)(z-\alpha_k)^{\nu+s_k-1} \right|.$$

Note that

$$\left(\frac{1-|\alpha_k|^2}{|1-\overline{\alpha_k} z|} \right)^{p+p_k+1} |z-\alpha_k|^{s_k-1+\nu} \leq \frac{(1-|\alpha_k|^2)^{p+p_k+1}}{|1-\overline{\alpha_k} z|^{p+p_k+2-s_k-\nu}} =$$
$$= \frac{(1-|\alpha_k|^2)^{p+1} \cdot (1-|\alpha_k|^2)^{p_k}}{|1-\overline{\alpha_k} z|^{p+p_k+2-s_k-\nu}} \leq \frac{(1-|\alpha_k|^2)^{p+1}}{|1-\overline{\alpha_k} z|^{p+2}}.$$

But

$$\frac{(1-|\alpha_k|^2)^{p+1}}{|1-\overline{\alpha_k} z|^{p+2}} \leq \frac{(1-|\alpha_k|^2)^{p+1}}{(1-|z|)^{p+2}}.$$

Therefore

$$|\varphi(z)| \leq c_0 \sum_{k=1}^{+\infty} |\widetilde{\Omega}_k(z)| \leq c_0 \sum_{k=1}^{+\infty} |g_{\alpha+1}(z)| \frac{|\pi_{p,k}(z)|}{(s_k-1)!} \cdot \frac{(1-|\alpha_k|^2)^{p+1}}{(1-|z|)^{p+2}}.$$

Taking into account a well-known estimate of M. M. Djrbashian's infinite product (see [9]):

$$\ln^+ |\pi_{p,k}(z, \alpha_j)| \leq c_p \sum_{j=1}^{+\infty} \left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_j z|} \right)^{p+1},$$

we get:

$$|\varphi(z)| \leq c_0 |g_{\alpha+1}(z)| \cdot \exp \left\{ c_p \sum_{j=1}^{+\infty} \left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_j z|} \right)^{p+1} \right\} \times$$
$$\times \frac{1}{(1-|z|)^{p+2}} \sum_{k=1}^{+\infty} \frac{1}{(s_k-1)!} (1-|\alpha_k|^2)^{p+1}.$$

From (2.2) it follows that (2.9) is valid and therefore

$$|\varphi(z)| \leq c_0 |g_{\alpha+1}(z)| \cdot \exp \left\{ c_p \sum_{j=1}^{+\infty} \left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_j z|} \right)^{p+1} \right\} \cdot \frac{c_3}{(1-|z|)^{p+2}}.$$

So the function f satisfies the following estimate:

$$|f(z)| \le c_0 |g_{\alpha+1}(z)|^2 \cdot \exp\left\{ c_p \sum_{j=1}^{+\infty} \left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_j z|} \right)^{p+1} \right\} \cdot \frac{c_3}{(1-|z|)^{p+2}}.$$

Using this estimate, we prove that the function f belongs to the class $S^{\infty}_{\alpha,a}$, that is

$$T(r,f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta \le \frac{C}{(1-r)^{\alpha}},$$

where $\alpha > 0$, $C > 0$.

We have:

$$T(r,f) \le const \cdot \sum_{j=1}^{+\infty} \int_{-\pi}^{\pi} \left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_j re^{i\theta}|} \right)^{p+1} d\theta + 2 \int_{-\pi}^{\pi} \ln^+ |g_{\alpha+1}(re^{i\theta})| d\theta +$$

$$+ 2\pi \ln \frac{c}{(1-r)^{p+2}}.$$

We estimate each term in this sum separately. As stated in [12] (see also [11]),

$$\sum_{j=1}^{+\infty} \int_{-\pi}^{\pi} \left(\frac{1-|\alpha_j|}{|1-\overline{\alpha}_j re^{i\theta}|} \right)^{p+1} d\theta \le \frac{c}{(1-r)^{\alpha}}, \tag{3.2}$$

for all $p > \alpha$.

Further, from (3.1) we have:

$$\int_{-\pi}^{\pi} \ln^+ |g_{\alpha+1}(re^{i\theta})| d\theta \le \sum_{s=1}^{n} \int_{-\pi}^{\pi} \frac{C d\theta}{|1-re^{i(\theta-\theta_s)}|^{\alpha+1}}.$$

Applying elementary estimates, we obtain:

$$\int_{-\pi}^{\pi} \ln^+ |g_{\alpha+1}(re^{i\theta})| d\theta \le \frac{c_3}{(1-r)^{\alpha}}. \tag{3.3}$$

We conclude from (3.2), (3.3) that the function f belongs to the class $S^{\infty}_{\alpha,a}$.

The converse statement follows directly from the result of Shamoyan and Rodikova (2014) (see [13]).

□

Proof. of Theorem 2.3.
Necessity. Suppose that there exists a function $F \in S^{\infty}_{\alpha}$ with the principal parts of the form (2.6), that is

$$F(z) = H(z, z_k, a_k) + \psi(z),$$

where the function ψ is holomorphic in some neighborhood of z_k, $k = 1, 2, \ldots$ For a given fixed $n \in \mathbb{N}$ we multiply this equality by the product of $\pi^n_p(z, z_j)$:

$$F(z) \cdot \pi^n_p(z, z_j) = H(z, z_k, a_k) \cdot \pi^n_p(z, z_j) + \psi(z) \cdot \pi^n_p(z, z_j).$$

It is obvious that the singular points of the function F are the members of the sequence $\{z_k\}$, so $F \cdot \pi_p^n \in H(D)$. Since $F \in S_\alpha^\infty$ and $\pi_p^n \in S_\alpha^\infty$ for all $p > \alpha$, we have $F \cdot \pi_p^n \in S_\alpha^\infty$. Thus

$$T(r, F \cdot \pi_p^n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |F(re^{i\theta}) \cdot \pi_p^n(re^{i\theta}, z_j)| d\theta \leq \frac{C(n)}{(1-r)^\alpha},$$

where $\alpha > 0$, $p > \alpha$, $C > 0$.

We denote $\Phi_n = F \cdot \pi_p^n$. We find the value of the function Φ in the points of the sequence $\{z_k\}$:

$$\Phi_n(z_k) = \lim_{z \to z_k} \Phi_n(z) = \lim_{z \to z_k} F(z) \cdot \pi_p^n(z, z_j) =$$

$$= \lim_{z \to z_k} \left(\frac{a_{k,n}}{(z-z_k)^n} + \frac{a_{k,n-1}}{(z-z_k)^{n-1}} \ldots + \frac{a_{k,1}}{(z-z_k)} \right) \cdot \pi_n^p(z, z_j) + \psi(z) \cdot \pi_p^n(z, z_j) =$$

$$= \lim_{z \to z_k} \frac{a_{k,n} \cdot \pi_p^n(z, z_j)}{(z-z_k)^n} = a_{k,n} \cdot {\pi'}_p^n(z_k).$$

Whence,

$$a_{k,n} = \frac{\Phi_n(z_k)}{{\pi'}_p^n(z_k)}.$$

Since $\Phi_n \in S_\alpha^\infty$, we have:

$$|\Phi_n(z_k)| \leq \exp \frac{c}{(1-|z_k|)^{\alpha+1}},\ k = 1, 2, \ldots \tag{3.4}$$

As stated in [13] (see Theorem A), if $\{z_k\}_1^\infty \in \widetilde{\Delta}$, then the following estimate is valid:

$$|{\pi'}_p^n(z_k, z_j)| \geq \exp \frac{-M}{(1-|z_k|)^{\alpha+1}}, \tag{3.5}$$

for all $k = 1, 2, \ldots$

Taking into account the estimate (3.4), (3.5), we obtain:

$$|a_{k,n}| \leq \exp \frac{c_0(n)}{(1-|z_k|)^{\alpha+1}},\ i = \overline{1, n}. \tag{3.6}$$

We prove the similar estimates on the coefficients $a_{k,i}$ for all $i < n$. Represent the function F as

$$F(z) = \sum_{l=1}^{n} \frac{a_{k,l}}{(z-z_k)^l} + \psi(z).$$

We have

$$F(z) - \sum_{\substack{l=1 \\ l \neq i}}^{n} \frac{a_{k,l}}{(z-z_k)^l} = \frac{a_{k,i}}{(z-z_k)^i} + \psi(z).$$

Multiply this equality by the product of $\pi_p^n(z,z_j)$:

$$\Phi_i(z)=F(z)\cdot\pi_p^n(z,z_j)-\sum_{\substack{l=1\\ l\neq i}}^{n}\frac{a_{k,l}}{(z-z_k)^l}\cdot\pi_p^n(z,z_j)=$$

$$=\frac{a_{k,i}}{(z-z_k)^i}\cdot\pi_p^n(z,z_j)+\psi(z)\cdot\pi_p^n(z,z_j).$$

Obviously, $\Phi_i\in S^\infty_{\alpha,a}$.

For a fixed $i\in\mathbb{N}$ we denote $\Psi_i(z)=\frac{\Phi_i(z)}{(z-z_k)^{n-i}}$, $z\in D$:

$$\Psi_i(z)=F(z)\cdot\frac{\pi_p^n(z,z_j)}{(z-z_k)^{n-i}}-\sum_{l=1}^{n}\frac{a_{k,l}}{(z-z_k)^l}\cdot\frac{\pi_p^n(z,z_j)}{(z-z_k)^n}. \tag{3.7}$$

It is clear that $\Psi_i\in S^\infty_{\alpha,a}$. Find the value of the function Ψ_i in the points of the sequence $\{z_k\}$:

$$\Psi_i(z_k)=\lim_{z\to z_k}\Psi_i(z)=\lim_{z\to z_k}\frac{a_{k,i}}{(z-z_k)^n}\cdot\pi_p^n(z,z_j)+\psi(z)\cdot\frac{\pi_p^n(z,z_j)}{(z-z_k)^{n-i}}=$$
$$=a_{k,i}\cdot{\pi'}_p^n(z_k).$$

Whence,

$$a_{k,i}=\frac{\Psi_i(z_k)}{{\pi'}_p^n(z_k)}.$$

Since $\Psi_i\in S^\infty_{\alpha,a}$, we have

$$|\Psi_i(z_k)|\le\exp\frac{c}{(1-|z_k|)^{\alpha+1}},\ k=1,2,...,$$

where $c\neq c(i)$, as can be seen from the representation (3.7).

Applying estimate (3.5), we obtain:

$$|a_{k,i}|\le\exp\frac{c_1(n)}{(1-|z_k|)^{\alpha+1}},\ i<n. \tag{3.8}$$

Sufficiency. The proof is by induction on i. For $i=1$ we denote

$$c_{k,1}=a_{k,1}\cdot{\pi'}_p^n(z_k,z_j),\ k=1,2,...$$

By the induction hypothesis,

$$|a_{k,1}|\le\exp\frac{c}{(1-|z_k|)^{\alpha+1}}. \tag{3.9}$$

Further, as stated in [10], the class S^∞_α is invariant under the differentiation operator, so ${\pi'}_p\in S^\infty_{\alpha,a}$ for all $p>\alpha$, that means the following estimate is valid:

$$|{\pi'}_p^n(z_k,z_j)|\le\exp\frac{c_1}{(1-|z_k|)^{\alpha+1}}. \tag{3.10}$$

Integrating the estimates (3.9), (3.10), we obtain:

$$|c_{k,1}|\le\exp\frac{c_2}{(1-|z_k|)^{\alpha+1}},\ k=1,2,...$$

By Theorem A there exists a function $\Phi \in S^{\infty}_{\alpha,a}$, such that $\Phi(z_k) = c_{k,1}$, $k = 1, 2, ...$

Let $F(z) = \frac{\Phi(z)(z-z_k)^{n-1}}{\pi^n_p(z,z_j)}$. Obviously, $F \in S^{\infty}_{\alpha}$. Function F has simple poles in the points of the sequence $\{z_k\}^{+\infty}_{k=1}$. Therefore the function F can be expressed as

$$F(z) = \frac{a'_{k,1}}{(z - z_k)} + \psi(z),$$

where ψ is holomorphic function in certain neighborhood of the point z_k, $k = 1, 2, \dots$

Find the coefficients $a'_{k,1}$:

$$a'_{k,1} = \lim_{z \to z_k} (F(z) - \psi(z))(z - z_k) = \frac{\Phi(z_k)}{\pi'^n_p(z_k, z_j)}.$$

But $\Phi(z_k) = c_{k,1}$, so $a'_{k,1} = a_{k,1}$. It means that for $i = 1$ the function F is required.

Now we assume the validity of the theorem in case $i = n - 1$. It means that there exists a function F_1, such that

$$F_1(z) = \frac{a''_{k,n-1}}{(z - z_k)^{n-1}} \dots + \frac{a''_{k,1}}{(z - z_k)} + \psi(z),$$

where ψ is holomorphic function in certain neighborhood of the point $z = z_k$, and a Laurent coefficients satisfy (2.7).

We prove the statement of the theorem for $i = n$. Let

$$c_{k,n} = a_{k,n} \cdot \pi'^n_p(z_k, z_j),\ k = 1, 2, ... \tag{3.11}$$

By hypothesis, we have:

$$|a_{k,n}| \le \exp \frac{c_2}{(1 - |z_k|)^{\alpha+1}}.$$

As stated in Theorem A, there exists a function $\Phi \in S^{\infty}_{\alpha,a}$, such that $\Phi(z_k) = c_{k,n}$, $k = 1, 2, ...$

Let $F(z) = \frac{\Phi(z)}{\pi^n_p(z,z_j)}$. Since

$$T(r, F) \le T(r, \Phi) + T\left(r, \frac{1}{\pi^n_p}\right),$$

then taking into account that function Φ belongs to the class $S^{\infty}_{\alpha,a}$ and the Nevanlinna equality of the balance, we conclude that $F \in S^{\infty}_{\alpha}$.

Function F has poles of order n in the points of the sequence $\{z_k\}^{+\infty}_{k=1}$. Therefore the function F can be expressed as

$$F(z) = \frac{\Phi(z)}{\pi^n_p(z)} = \frac{a'_{k,n}}{(z - z_k)^n} + \frac{a'_{k,n-1}}{(z - z_k)^{n-1}} + \dots + \frac{a'_{k,1}}{(z - z_k)} + \psi(z),$$

where ψ is holomorphic function around the point z_k. Find the coefficients $a'_{k,n}$, $k = 1, 2, \dots$

$$a'_{k,n} = \lim_{z \to z_k} F(z)(z - z_k)^n = \lim_{z \to z_k} \frac{(z - z_k)^n \cdot \Phi(z)}{\pi_p^n(z, z_j)} = \frac{\Phi(z_k)}{\pi_p'^n(z_k)}.$$

Using that $\Phi(z_k) = c_{k,n}$, $k = 1, 2, \dots$ and based on (22) we conclude: $a'_{k,n} = a_{k,n}$,$k = 1, 2, \dots$ We write the expansion of the function Φ in the Taylor series at the point $z = z_k$, $k = 1, 2, \dots$,

$$\Phi(z) = \Phi(z_k) + \sum_{j=1}^{+\infty} \frac{\Phi^{(j)}(z_k) \cdot (z - z_k)^j}{j!}$$

On the other hand,

$$\Phi(z) = \frac{a'_{k,n}}{(z - z_k)^n} \pi_p^n(z, z_j) + \dots + \frac{a'_{k,1}}{(z - z_k)} \pi_p^n(z, z_j) + \psi(z) \pi_p^n(z, z_j).$$

Thus, using the latest representation for Φ, we have

$$\Phi(z_k) - \frac{a'_{k,n}}{(z - z_k)^n} \pi_p^n(z, z_j) + \frac{\Phi'(z_k) \cdot (z - z_k)}{1!} + \frac{\Phi''(z_k) \cdot (z - z_k)^2}{2!} + \dots =$$

$$= \frac{a'_{k,n-1}}{(z - z_k)^{n-1}} \pi_p^n(z, z_j) + \dots + \frac{a'_{k,1}}{(z - z_k)} \pi_p^n(z, z_j) + \psi(z) \pi_p^n(z, z_j).$$

Divide both sides of this equality by the expression $(z - z_k)$:

$$\left(\Phi(z_k) - \frac{a'_{k,n}}{(z - z_k)^n} \pi_p^n(z, z_j) \right) \cdot \frac{1}{(z - z_k)} + \frac{\Phi'(z_k)}{1!} + \frac{\Phi''(z_k) \cdot (z - z_k)}{2!} + \dots =$$

$$= \frac{a'_{k,n-1}}{(z - z_k)^n} \pi_p^n(z, z_j) + \dots + \frac{a'_{k,1}}{(z - z_k)^2} \pi_p^n(z, z_j) + \frac{\psi(z) \pi_p^n(z, z_j)}{(z - z_k)}.$$

Limiting $z \to z_k$, we get:

$$\frac{\Phi'(z_k)}{1!} = a'_{k,n-1} \cdot \pi'^n_p(z_k).$$

In a similar way we find that

$$a'_{k,i} \cdot \pi'^n_p(z_k) = \frac{\Phi^{(n-i)}(z_k)}{(n - i)!}, \; i < n.$$

Since the class S_α^∞ is invariant under the differentiation operator, we obtain

$$|a'_{k,i}| \leq \exp \frac{c}{(1 - |z_k|)^{\alpha+1}}, \; i = 1, 2, \dots, n - 1$$

But

$$|a_{k,i}| \leq \exp \frac{c_0}{(1 - |z_k|)^{\alpha+1}}, \; i = 1, 2, \dots, n - 1,$$

hence

$$|a_{k,i} - a'_{k,i}| \leq \exp \frac{c_1}{(1 - |z_k|)^{\alpha+1}}, \; i = 1, 2, \dots, n - 1.$$

By the induction hypothesis, there exists a function F_1, expressed in the form

$$F_1(z) = \frac{a''_{k,n-1}}{(z-z_k)^{n-1}} + \ldots + \frac{a''_{k,1}}{(z-z_k)} + \psi_1(z),$$

where $a''_{k,i} = a_{k,i} - a'_{k,i}$, $i = 1, 2, \ldots, n-1$, $\psi_1(z)$ is analytic function in some neighborhood of the point $z = z_k$, $k = 1, 2, \ldots$ Consider the function $F + F_1 \in S^{\infty}_{\alpha}$. Based on the above, we may conclude that the function $F + F_1$ has the following principal parts:

$$H(z, z_k, a_k) = \frac{a_{k,n}}{(z-z_k)^n} + \ldots + \frac{a_{k,1}}{(z-z_k)}, \; k = 1, 2, \ldots$$

□

References

[1] Carleson, L.,*An interpolation problem for bounded analytic functions*, Amer. J. Math. 80,(1958), 921–930 .

[2] Djrbashian, M.M., *On the representation problem of analytic functions*, Soob. Inst. Math. i Mekh. AN ArmSSR. 2,(1948), 3-40 . (in Russian)

[3] Djrbashian, M.M., *Basicity of some biorthogonal systems and the solution of the multiple interpolation problem in H^p-classes in the halfplane*, Izv. Akad. Nauk ArmSSR, Matematika. 43, no 6,(1978), 1327 - 1384 . (in Russian)

[4] Goldberg, A.A., Ostrovsky, I.V. *Value distribution of meromorphic functions*, Moscow, Nauka,(1970). (in Russian)

[5] Hartmann, A., Massaneda, X., Nicolau, A., Thomas, P., *Interpolation in the Nevanlinna and Smirnov classes and harmonic majorants*, J. Funct. Anal. 217,(2004), 1–37 .

[6] Naftalevic, A.G., *On interpolation by functions of bounded characteristic*, Vilniaus Valst. Univ. Mokslu Darbai. Mat. Fiz. Chem. Mokslu Ser. 5,(1956), 5–27 . (in Russian)

[7] Nevanlinna, R., *Eindeutige analytische Funktionen*, 2nd ed. Springer-Verlag, Berlin (1953).

[8] Seip, K., *Interpolation and sampling in spaces of analytic functions*. University Lecture Series, Amer. Math. Soc., Providence. 33 (2004).

[9] Shamoyan, F.A., *M. M. Dzhrbashians factorization theorem and characterization of zeros of functions analytic in the disk with a majorant of bounded growth*, Izv. Akad. Nauk ArmSSR, Matematika. 13, no 5-6, (1978), 405-422 . (in Russian)

[10] Shamoyan, F.A., Bednazh, V.A., *Multiple interpolation in weighted classes of analytic functions in the disk*, Siberian Electronic Mathematical Reports. 11, (2014), 354-361 . (in Russian)

[11] Shamoyan, F.A., Shubabko, E.N., *Introduction to the theory of weighted L^p-classes of meromorphic functions*. Bryansk State University, Bryansk (2009). (in Russian)

[12] Shamoyan, F.A., Shubabko, E.N., *Parametrical representations of some classes of holomorphic functions in the disk*, Operator Theory: Advances and Applications, Birkhauser Verlag, Basel. 113, (2000), 331–338 .

[13] Shamoyan, F.A., Rodikova, E.G., *On interpolation in the class of analytic functions in the unit disk with power growth of the Nevanlinna characteristic*, Journal of Siberian Federal University, Math. and phys. 7,no 2,(2014), 235–243.

[14] Shapiro, J., Shields, A.,*On some interpolation problems for analytic functions*, Amer. J. Math. 83,(1961), 513–532 .

[15] Stein, E.M., *Singular integrals and differentiability properties of functions*, Princeton University Press (1970).

[16] Tsuji, M. ,*Potential theory in modern function theory*, Tokyo, (1959).

[17] Tsuji, M., *Canonical product for a meromorphic functions in a unit circle*, J. Math. Jap. , Vol. 8, no 1,(1956), 7-19 .

[18] Vinogradov, S.A., Havin, V.P., *Free interpolation in H^∞ and in some other classes of functions*, J. of Math. Sci. 47, (1974), 15–54 . (in Russian)

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