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Fourier Transform and Quasi-analytic Classes of Functions of Bounded Type on Tubular Domains*

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Abstract. A condition for a function of bounded type to belong to the Hardy class H^1 in terms of the Fourier transform of the boundary values of this function on $\mathbb{R}^n$ is found. Applications of the obtained result to the theories of Hardy classes and of quasi-analytic classes of functions are given.

Key words: Fourier transform, quasi-analytic classes, pluriharmonic function, tubular domain.

1. Let

$$\mathbb{C}^n_+ = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Im} z_j > 0,\ j = 1, \dots, n\},$$

and let

$$N(\mathbb{C}^n_+) = \{f : f(z) = h_1(z)/h_2(z),\ h_j(z) \in H^\infty(\mathbb{C}^n_+),\ j = 1, 2,\ h_2(z) \neq 0,\ z \in \mathbb{C}^n_+\}$$

be the class of functions of bounded type on $\mathbb{C}^n_+$. In the one-dimensional case, the class $N(\mathbb{C}_+)$ coincides with the Nevanlinna class in the upper half-plane $\mathbb{C}_+ := \mathbb{C}^1_+$ (see [1, Sec. II.5]); for $n \geqslant 2$, the Nevanlinna class, i.e., the class of functions f for which $\ln|f|$ has an n-harmonic majorant, is fundamentally different from $N(\mathbb{C}^n_+)$ (see [2, Theorem 4.4.1], [3, p. 165]). This can be seen from the following simple considerations: in the one-dimensional case, the zero sets of bounded analytic functions and functions in the Nevanlinna class both on the unit disk U and in the half-plane are described by Blaschke's classical condition. In the multidimensional case ($n \geqslant 2$), for any p, $0 < p < +\infty$, there exists a nontrivial function f in the Hardy class $H^p(U^n)$ on the unit polydisk U^n such that its zero set $Z_f = \{z : f(z) = 0\}$ is a uniqueness set for the class of bounded analytic functions on U^n (see [2, p. 61]). This is also true for $H^\infty(\mathbb{C}^n_+)$. Obviously, the corresponding function f belongs to the Nevanlinna class but does not belong to the class of analytic functions of bounded type on $\mathbb{C}^n_+$. In the same way, by using a property of the zero sets of functions in the Hardy classes on the ball (see [3, p. 165]), it can be shown that the class of functions of bounded type differs from the Nevanlinna class on the ball. As is known, if f belongs to the Smirnov class $N^+(\mathbb{C}^n_+)$ (see [2, Theorem 3.3.5] and [4, p. 246]) and its boundary value on $\mathbb{R}^n$ belongs to $L^1(\mathbb{R}^n)$, then f belongs to the Hardy class $H^1(\mathbb{C}^n_+)$ and, therefore, the Fourier transform $\hat{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(t) e^{-itx}\, dt$ of this function vanishes on $\mathbb{R}^n \setminus \mathbb{R}^n_+$. The simple example of the function

$$f_a(z) = \prod_{j=1}^{n} (i + z_j)^{-s} \exp(-i a_j z_j),$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n_+$, $a = (a_1, \dots, a_n) \in \mathbb{R}^n_+$, and $s > 1$, shows that this assertion does not hold for $N(\mathbb{C}^n_+)$.

For $n = 1$, it was proved in [5] that if $\hat{f}(x) \to 0$ sufficiently rapidly as $x \to -\infty$, then the function $\hat{f}$ identically vanishes on $\mathbb{R}_-$. Moreover, a necessary and sufficient condition on the rate of decrease for this assertion to hold was found.

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A substantial role in the proof of this result was played by a factorization of functions in $N(\mathbb{C}_+)$ (see [1, Sec. II.5, (5.10)]). It is well known that, for $n \geqslant 2$, there are no factorizations (see [2, Theorem 4.4.1]). In this paper, we state a similar assertion for all $n \geqslant 1$; its proof does not use factorization. We also give several interesting, in our view, applications to the theories of Hardy classes and of quasi-analytic classes of functions.

2. In what follows, we use the following standard notation (see, e.g., [3] and [4]). For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we set $\|z\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$, and for $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we set $|k| = k_1 + \dots + k_n$. Let $M(r) = M(r_1, \dots, r_n)$, where $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, be a positive function monotonically increasing with respect to each variable $r_j \in \mathbb{R}_+$, $1 \leqslant j \leqslant n$, for fixed $r' = (r_1, \dots, r_{j-1}, r_{j+1}, r_n) \in \mathbb{R}_+^{n-1}$ and satisfying the condition

$$\lim_{\|r\| \to +\infty} \frac{\ln \|r\|}{\ln M(r)} = 0.$$

We set

$$M_m^{(j)} = \sup_{x \in \mathbb{R}_+} \frac{x^m}{M(r_1, \dots, r_{j-1}, x, r_{j+1}, \dots, r_n)}, \qquad m \in \mathbb{Z}_+,\ j = 1, \dots, n,$$

$$T_j(t) = \sup_{m \geqslant 1} \frac{t^m}{M_m^{(j)}}, \qquad t \in \mathbb{R}_+ .$$

By $\mathbb{RP}(\mathbb{R}^n)$ we denote the class of functions $f \in L^1(\mathbb{R}^n)$ for which the Cauchy-type integral identically vanishes on $\mathbb{C}^n \setminus \{\mathbb{C}_+^n \cup \mathbb{C}_-^n\}$ if $n \geqslant 2$ and on $L^1(\mathbb{R})$ if $n = 1$.

Theorem 1. *Suppose that $f \in N(\mathbb{C}_+^n)$, the boundary values of f on $\mathbb{R}^n$ belong to the class $\mathbb{RP}(\mathbb{R}^n)$, and*

$$\overline{\lim_{\|y\| \to +\infty}} \frac{\ln |f(iy)|}{\|y\|} \leqslant 0. \tag{1}$$

Suppose also that the Fourier transform of the function f is bounded as

$$|\hat{f}(-x)| \leqslant \frac{1}{M(x)}, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}_+^n. \tag{2}$$

If, in addition,

$$\int_1^{+\infty} \frac{\ln T_j(r)}{r^{3/2}}\, dr = +\infty, \qquad j = 1, \dots, n, \tag{3}$$

then $\hat{f}(x) = 0$ for all $x \in \mathbb{R}^n \setminus \mathbb{R}_+^n$, and f belongs to the Hardy class $H^1(\mathbb{C}_+^n)$. Conversely, if

$$M(x_1, \dots, x_n) = \exp\Bigl(\sum_{j=1}^n P_j(x_j)\Bigr), \qquad \frac{P_j'(t)t}{P_j(t)} \nearrow +\infty \quad \text{as } t \to +\infty,\ j = 1, \dots, n, \tag{4}$$

and at least one of the integrals in (3) converges, then there exists a function $f \in N(\mathbb{C}_+^n) \cap \mathbb{RP}(\mathbb{R}^n)$ such that $\hat{f}$ satisfies (2) but $\hat{f}(x) \neq 0$ for $x \in \mathbb{R}_-^n$ if $f \notin H^1(\mathbb{C}_+^n)$.

Remark 1. For the case where the majorant M has the form (4), the author proved Theorem 1 by a different method in his previous paper [6]. We also mention that Theorem 1 is essentially used in the proofs of Theorems 2 and 3.

Remark 2. The example of the function $f(z_1, z_2) = \frac{\varphi(z_1)}{(i+z_1)^2(i+z_2)^2 S(z_2)}$, $z = (z_1, z_2) \in \mathbb{C}_+^2$, where $\varphi \in H^\infty(\mathbb{C}_+)$ and S is any inner function in $\mathbb{C}_+$, shows that the condition that the boundary values of f belong to the class $\mathbb{RP}(\mathbb{R}^n)$ is necessary for the validity of Theorem 1.

It follows from the form of the function f_a that there exists a function $g \in C^\infty(\mathbb{R}^n) \cap N(\mathbb{C}_+^n) \cap L^1(\mathbb{R}^n)$ such that $\lim_{|x| \to +\infty} \partial^k g(x)/\partial x^k = 0$ for any $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and $\int_{\mathbb{R}^n} |g(x+iy)|dx \geqslant c_0 \exp(a \cdot y)$, $a = (a_1, \dots, a_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}_+^n$, $c_0 > 0$. Nevertheless, the following theorem is valid.

Theorem 2. *Suppose that* $M = \{M_s\}_{s=1}^{+\infty}$ *is a monotonically increasing sequence of positive numbers,*

$$C^\infty(M) = \left\{ g \in C^\infty(\mathbb{R}^n) : \left| \frac{\partial^{|k|} g(x_1, \ldots, x_n)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right| \leqslant A^{|k|} M_{|k|}, x \in \mathbb{R}^n \right\},$$

$$T(r) = \sup_{s \geqslant 1} \frac{r^s}{M_s}, \qquad r \in \mathbb{R}_+, \tag{5}$$

and

$$f \in N(\mathbb{C}_+^n) \cap \mathbb{R}\mathrm{P}(\mathbb{R}^n) \cap \mathbb{C}^\infty(M). \tag{6}$$

If condition (1) *holds and*

$$\int_1^{+\infty} \frac{\ln T(r)}{r^{3/2}} dr = +\infty, \tag{7}$$

then $f \in H^1(\mathbb{C}_+^n)$ *and* $\int_{\mathbb{R}^n} |f(x+iy)| dx \leqslant \int_{\mathbb{R}^n} |f(x)|\, dx$ *for all* $y \in \mathbb{R}_+^n$.

Conversely, if the integral (7) *converges or condition* (1) *is violated, then there exists a function* f *satisfying* (6) *but not belonging to the class* $H^1(\mathbb{C}_+^n)$.

The following theorem refines Salinas' classical theorem [7] for $n = 1$.

Theorem 3. *Suppose that* $g \in C^\infty(M)$ *and there exists a function* $f \in N(\mathbb{C}_+^n) \cap \mathbb{R}\mathrm{P}(\mathbb{R}^n)$ *satisfying condition* (1) *and such that* $\lim_{|y| \to 0} f(x+iy) = g(x)$ *almost everywhere on* $\mathbb{R}^n$. *Suppose also that* $T(r)$ *is defined by* (5), *provided that condition* (7) *holds, and*

$$\frac{\partial^{|k|} g(x_0)}{\partial x^k} = 0, \qquad k \in \mathbb{Z}_+^n, \tag{8}$$

for some $x_0 \in \mathbb{R}^n$. *Then* $g(x) = 0$ *for all* $x \in \mathbb{R}^n$.

Conversely, if the integral (7) *converges, then there exists a function* $g \in C_A^\infty(M) = C^\infty(\mathbb{C}_+^n \cup \mathbb{R}^n) \cap H(\mathbb{C}_+^n)$ *such that condition* (8) *holds but* g *does not identically vanish on* $\mathbb{R}^n$; *here* $H(\mathbb{C}_+^n)$ *denotes the set of all analytic functions in* $\mathbb{C}_+^n$.

As mentioned above, the proofs of Theorems 2 and 3 are based on Theorem 1, and the proof of the latter reduces to solving the weak invertibility problem for bounded analytic functions in Bergman-type weight spaces in $\mathbb{C}_+^n$. This problem is solved in these spaces by the method of weighted approximation by algebraic polynomials (see [8]).

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