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EXTENSION OF STARLIKE FUNCTIONS TO A FINITELY PUNCTURED PLANE

In memory of Alexander Vasil'ev

Abstract. We consider a sequence of functions which are starlike in the unit disk and their logarithmic derivatives are meromorphic with a finite number of simple poles in any boundary domain. These poles are either boundary deterministic or random with given characteristics. The aim of the article is the limit process and properties of the limit functions. We distinguish conditions for residues and distribution of poles. Under certain conditions, the sequence converges to the identity function. Another conditions allow us to obtain estimates for the limit function and its logarithmic derivative.

Key words: *starlike function, meromorphic function, weight, deterministic point regulation, uniform distribution*

2010 Mathematical Subject Classification: *30C45, 60G55*

1. Introduction. During last decades stochastic ideas and methods became quite popular in geometric function theory. Among the most known events we mention that in 2000 Schramm introduced in [1] a family of random curves, which he called stochastic Loewner evolution, or SLE, see also [2] for further development.

In this article we present an approach to a (partly) stochastic problem posed by Vasil'ev for functions which are holomorphic in the plane, except for at most a countable set of singular points, and are starlike on the disk.

A function f is said to be starlike if it maps the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto domain $f(\mathbb{D})$ which is starlike with respect to the origin. A holomorphic function f on $\mathbb{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$ is starlike if and only if the real part of $zf'(z)/f(z)$ is positive in $\mathbb{D}$ (see, e.g., [3, Theorem 2.10]). A dense subclass $\{f_n\}_{n\geq 1}$ of the set S^* of all starlike functions f is defined by

$$\frac{zf_n'(z)}{f_n(z)} = \sum_{k=1}^{n} \mu_{kn} \frac{a_k + z}{a_k - z}, \quad z \in \mathbb{D}, \tag{1}$$

where $|a_k| = 1$, $0 < \mu_{kn} < 1$, $k = 1, \dots, n$, $\sum_{k=1}^{n} \mu_{kn} = 1$, $n \in \mathbb{N}$.

In recent papers Alexander Vasil'ev developed stochastic topics in complex analysis, see, e.g., [4–6]. In early 2014 Vasil'ev proposed the problem of studying a limit process $f_n \to f$ as $n \to \infty$ when every starlike function f_n is given by (1) with $|a_k| \geq 1$, $k = 1, \dots, n$, and $a_n \to \infty$ as $n \to \infty$.

Surely, we have to decide which conditions are required for a sequence of singular points $\{a_n\}_{n\geq 1}$. It seems interesting to consider the Poisson configuration of random points $\{a_n\}$, see, e.g., [7, § 3.1].

Definition 1. *The* **spatial Poisson process** *with uniform intensity* $\beta > 0$ *is a point process in* $\mathbb{R}^2$ *that is a random distribution of points such that*

[PP1] *for every bounded closed set* B*, the count* $N(B)$ *which is the number of points of the process contained in* B *has a Poisson distribution with the mean* $\beta s(B)$*, where* $s(B)$ *denotes the area of* B*;*

[PP2] *if* $B_1, \dots, B_m$ *are disjoint regions, then* $N(B_1), \dots, N(B_m)$ *are independent.*

The Poisson process assumption relates only to the poles $\{a_k\}_{1\leq k\leq n}$ in (1). However, there are families of coefficients $\{\mu_{kn}\}_{1\leq k\leq n}$, $n \geq 1$, which are to be defined as well. We will prefer a deterministic way of choosing these weights. Let us describe two scenarios of weights behavior.

Given a sequence of numbers $\nu_1, \dots, \nu_n, \dots$, $0 < \nu_n \leq 1$ for all $n \geq 1$, denote

$$S_n = \sum_{k=1}^{n} \nu_k, \quad \mu_{kn} = \frac{\nu_k}{S_n}, \quad 1 \leq k \leq n, \quad n \geq 1. \tag{2}$$

For every $n \geq 1$, we obtain a set of n positive numbers $\mu_{1n}, \dots, \mu_{nn}$, $\sum_{k=1}^{n} \mu_{kn} = 1$. There are only two possibilities:

(i) $S_n \to \infty$ as $n \to \infty$;

(ii) $\lim_{n\to\infty} S_n = S < \infty$.

In scenario (i), all weights μ_{kn}, $1 \leq k \leq n$, $n \geq 1$, tend to 0 uniformly as $n \to \infty$. This makes it sensible to call $\nu_1, \dots, \nu_n, \dots$ in (i) a *uniform* *type sequence.* For example $(\nu_1, \dots, \nu_n, \dots) = (1, \dots, 1, \dots)$ is a uniform type sequence. Indeed, in this case, $S_n = n$ for all $n \geq 1$, and

$$\mu_{kn} = \frac{1}{n}, \quad 1 \leq k \leq n, \quad n \geq 1.$$

On the other hand, in scenario (ii) no one of weights μ_{kn}, $n \geq 1$, tends to 0 for a fixed k, $1 \leq k \leq n$, and $n \to \infty$, but μ_{kn} tend to 0 as $n \to \infty$ and $k \to \infty$, $1 \leq k \leq n$. In this case, we call $\nu_1, \dots, \nu_n, \dots$ in (ii) a *Poisson type sequence.* For example,

$$(\nu_1, \dots, \nu_n, \dots) = \left(1, \lambda, \dots, \frac{\lambda^{n-1}}{(n-1)!}, \dots\right), \quad \lambda > 0,$$

is a Poisson type sequence. Indeed, we see that $\lim\limits_{n\to\infty} S_n = e^{\lambda}$ and therefore, for any fixed k,

$$\lim_{n\to\infty} (\mu_{1n}, \dots, \mu_{kn}) = \left(e^{-\lambda}, e^{-\lambda}\lambda, \dots, e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}\right).$$

This is a sequence of values of a probability mass function of a discrete random variable having a Poisson distribution.

The article is organized as follows.

In Section 2 we deal with a deterministic point regulation, i.e., the points $\{a_n\}$ are determined and the weights $\{\mu_{kn}\}$ are generated by a sequence either of the uniform type or the Poisson type. We prove Theorem 1, which states that $f_n \to \text{id}$ as $n \to \infty$ in the first case, and Theorem 2 with estimates for the limit functions in the second case.

In Section 3 we replace axioms **[PP1]**–**[PP2]** by other ones which make the problem partly stochastic. Under the conditions of Theorem 3, functions f_n converge almost surely to the identity.

2. Deterministic point regulation. Let f_n, $n \geq 1$, be given by (1) and denote

$$p_n(z) := \frac{z f_n'(z)}{f_n(z)} = 1 + \sum_{j=1}^{\infty} p_{jn} z^j, \quad z \in \mathbb{D}. \tag{3}$$

We suppose that $\nu_1, \dots, \nu_n, \dots$ generating weights $\{\mu_{kn}\}$ is a uniform type sequence and show that $f_n \to \text{id}$ if $a_n \to \infty$ as $n \to \infty$.

Theorem 1. *Given $\{a_n\}_{n\geq 1}$ with $|a_n| \geq 1$ and $a_n \to \infty$ as $n \to \infty$, let f_n, $n \geq 1$, be represented by (3) where the weights $\{\mu_{kn}\}_{1\leq k\leq n<\infty}$ are* *generated according to (2) by a uniform type sequence $\nu_1, \dots, \nu_n, \dots$. Then f_n converge to the identity function as $n \to \infty$ uniformly on compact subsets of $\mathbb{D}$.*

Proof. Formulas (1) and (3) imply that coefficients p_{jn} in (3) are evaluated as

$$p_{jn} = 2\sum_{k=1}^{n} \frac{\mu_{kn}}{a_k^j}, \quad j \geq 1.$$

For any $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$|a_n| > 1 + \frac{1}{\epsilon}, \quad n > n_1,$$

and there exists $n_2 > n_1$ for which

$$0 < \mu_{kn} < \frac{\epsilon}{n_1}, \quad n > n_2.$$

Hence, for $n > n_2$ and any $j \geq 1$, we have

$$|p_{jn}| \leq 2\sum_{k=1}^{n_1} \frac{\mu_{kn}}{|a_k^j|} + 2\sum_{k=n_1+1}^{n} \frac{\mu_{kn}}{|a_k^j|} \leq$$

$$\leq 2\sum_{k=1}^{n_1} \mu_{kn} + 2\sum_{k=n_1+1}^{n} \mu_{kn} \frac{\epsilon}{1+\epsilon} < 2\epsilon + 2\epsilon = 4\epsilon.$$

This means that, for any $j \geq 1$, p_{jn} tend to 0 uniformly with respect to j as $n \to \infty$, and thus $\lim_{n\to\infty} p_n(z) = 1$. So $f_n \to \mathrm{id}$ as $n \to \infty$ which completes the proof of Theorem 1. □

Remark 1. *If $|a_n| = 1 + \beta n$, $\beta > 0$, $n \geq 1$, then, for f_n and p_n in (3) under conditions of Theorem 1, the following relations*

$$|p_n(z) - 1| = |z| O\left(\frac{1}{\sqrt{S_n}}\right), \quad \left|\log \frac{f_n(z)}{z}\right| = O\left(\frac{1}{\sqrt{S_n}}\right), \quad z \in \mathbb{D}, \quad n \to \infty,$$

hold.

Indeed,

$$|p_n(z) - 1| = \left|\sum_{k=1}^{n} \frac{2\mu_{kn} z}{a_k - 1}\right| \leq \sum_{k=1}^{n} \frac{2\mu_{kn}|z|}{\beta k} = \sum_{k=1}^{n} \frac{2\nu_k |z|}{\beta k} \left[\sum_{k=1}^{n} \nu_k\right]^{-1} \leq$$

$$\leq 2|z|\sqrt{\sum_{k=1}^{n}\nu_k^2}\sqrt{\sum_{k=1}^{n}\frac{1}{(\beta k)^2}}\left[\sum_{k=1}^{n}\nu_k\right]^{-1}\leq$$

$$\leq 2|z|\sqrt{\sum_{k=1}^{n}\frac{1}{(\beta k)^2}}\sqrt{\left[\sum_{k=1}^{n}\nu_k\right]^{-1}}\leq\frac{C|z|}{\sqrt{\sum_{k=1}^{n}\nu_k}}=\frac{C|z|}{\sqrt{S_n}}$$

with a certain constant $C>0$.

This implies that

$$\left|\log\frac{f_n(z)}{z}\right|=\left|\int\limits_0^z\frac{p_n(\zeta)-1}{\zeta}d\zeta\right|\leq\int\limits_0^z\frac{|p_n(\zeta)-1|}{|\zeta|}|d\zeta|\leq$$

$$\leq\int\limits_0^{|z|}\frac{C}{\sqrt{S_n}}d|\zeta|=\frac{C|z|}{\sqrt{S_n}}.$$

The situation is essentially different if $\nu_1,\ldots,\nu_n,\ldots$ generating weights $\{\mu_{kn}\}$ is a Poisson type sequence.

Theorem 2. *Given $\{a_n\}_{n\geq 1}$ with $|a_n|\geq 1$ and $a_n\to\infty$ as $n\to\infty$, let f_n, $n\geq 1$, be represented by (3) where the weights $\{\mu_{kn}\}_{1\leq k\leq n<\infty}$ are generated according to (2) by a Poisson type sequence $\nu_1,\ldots,\nu_n,\ldots$. Then $p=\lim\limits_{n\to\infty}p_n$ and $f=\lim\limits_{n\to\infty}f_n$ satisfy the following inequalities*

$$\frac{2}{S}\sum_{k=1}^{\infty}\nu_k\frac{|z|}{|a_k|+|z|}\leq|p(z)-1|\leq\frac{2}{S}\sum_{k=1}^{\infty}\nu_k\frac{|z|}{|a_k|-|z|},\quad z\in\mathbb{D},\tag{4}$$

$$|f(z)|\leq|z|\prod_{k=1}^{\infty}\left(1-\frac{|z|}{|a_k|}\right)^{-2\nu_k/S},\quad z\in\mathbb{D}.\tag{5}$$

Proof. Formulas (1)–(3) imply that

$$p(z)=\lim_{n\to\infty}p_n(z)=1+\lim_{n\to\infty}\sum_{j=1}^{\infty}p_{jn}z^j=1+\lim_{n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{n}\frac{2\mu_{kn}}{a_k^j}z^j=$$

$$=1+2\lim_{n\to\infty}\sum_{k=1}^{n}\mu_{kn}\sum_{j=1}^{\infty}\left(\frac{z}{a_k}\right)^j=1+2\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\nu_k}{S_n}\frac{z}{a_k-z}=$$

$$= 1 + \frac{2}{S}\sum_{k=1}^{\infty} \nu_k \frac{z}{a_k - z}.$$

From here we immediately deduce (4).

From the formula

$$\log \frac{f(z)}{z} = \int\limits_0^z \frac{p(\zeta) - 1}{\zeta} d\zeta$$

we find that

$$\log \left| \frac{f(z)}{z} \right| \leq \left| \log \frac{f(z)}{z} \right| \leq \int\limits_0^z \frac{2}{S}\sum_{k=1}^{\infty} \nu_k \frac{|d\zeta|}{|a_k| - |\zeta|} =$$

$$= \frac{2}{S}\sum_{k=1}^{\infty} \nu_k \int\limits_0^{|z|} \frac{d|\zeta|}{|a_k| - |\zeta|} = \frac{2}{S}\sum_{k=1}^{\infty} \nu_k \log \frac{|a_k|}{|a_k| - |z|} =$$

$$= \log \prod_{k=1}^{\infty} \left(1 - \frac{|z|}{|a_k|}\right)^{-2\nu_k/S}$$

which implies estimate (5) and completes the proof of Theorem 2. □

One can write down exact representations and estimates for the sequence $\nu_1, \ldots, \nu_n, \ldots$ with the Poisson distribution.

3. Random point regulation. In this section, we deal with a random model of point configurations based on the Poisson process of Definition 1. However, we will transform this model in accordance with the specific problem for starlike functions having random singular points in the plane.

First, we have seen in the previous sections that estimates of Theorems 1 and 2 are independent of $\arg a_n$. Therefore, it is natural to apply the one-dimensional Poisson process on the line, i.e., we assume that $a_n \geq 1$ for $n \geq 1$.

Secondly, we focus on a homogeneous Poisson point process which is also called a stationary Poisson process, see, e.g., [8, pp. 19–25], as well as uniform Poisson point process, see, e.g., [9, p.13]. If the homogeneous point process is defined on the real line in different mathematical models, then it has the characteristic that the positions will be uniformly distributed.

Third, a compound Poisson point process is formed by adding random values or weights to each point of the Poisson point process, so the process is constructed from a marked Poisson point process, where the marks form a collection of independent and identically distributed non-negative random values, see, e.g., [10, pp. 19–21; 11, pp. 291–293]. In our case, the weights $\{\mu_{kn}\}$ serve as deterministic marks either of the uniform type or the Poisson type.

Finally, we replace the axiom **[PP1]** in Definition 1 by a deterministic number of points in a given interval.

So we make the following assumptions.

[UP1] For every $R > 1$, there is a number $n(R)$ of points $\{a_k\}_{1 \le k \le n(R)}$ which arrive in the segment $[1, R]$.

[UP2] Random points $\{a_k\}_{1 \le k \le n(R)}$ are uniformly distributed in $[1, R]$, knowing the number $n(R)$.

[UP3] Random points $\{a_k\}$ are independent.

Theorem 3. *Suppose that a random point process satisfies axioms* **[UP1]** – **[UP3]**, *and let functions $f_{n(R)}$ be represented by (1) and (3) with arbitrary positive numbers $\{\mu_{kn(R)}\}_{1 \le k \le n(R)}$, $\sum\limits_{k=1}^{n(R)} \mu_{kn(R)} = 1$. Then $f_{n(R)}$ converge almost surely to the identity function $f(z) = z$ as $R \to \infty$.*

Proof. Formulas (1) and (3) imply that

$$p_{n(R)}(z) = 1 + \sum_{j=1}^{\infty} p_{jn(R)} z^j, \quad z \in \mathbb{D},$$

where

$$p_{jn(R)} = \sum_{k=1}^{n(R)} \frac{2\mu_{kn(R)}}{a_k^j}.$$

As far as random points $\{a_k\}_{1 \le k \le n(R)}$ are independent, a joint probability density function $\varphi(a_1, \dots, a_{n(R)})$ is evaluated as follows

$$\varphi(a_1, \dots, a_{n(R)}) = \prod_{k=1}^{n(R)} \varphi(a_k),$$

where $\varphi(a_k)$ is a probability density function for the random point a_k. Since a_k is uniformly distributed in $[1, R]$, its probability density function has the form

$$\varphi(a_k) = \frac{1}{R-1} \text{ for } a_k \in [1,R], \text{ and } \varphi(a_k) = 0 \text{ for } a_k \notin [1,R],$$

$k = 1, \dots, n(R)$.

Evaluate the mathematical expectation $\mathbb{E}(p_{jn(R)})$ of $p_{jn(R)}$, $j \geq 1$,

$$\mathbb{E}(p_{jn(R)}) = \int_1^R \dots \int_1^R \sum_{k=1}^{n(R)} \frac{2\mu_{kn(R)}}{a_k^j} \varphi(a_1, \dots, a_{n(R)}) da_1 \dots da_{n(R)} =$$

$$= \sum_{k=1}^{n(R)} 2\mu_{kn(R)} \int_1^R \frac{\varphi(a_k)}{a_k^j} da_k = \sum_{k=1}^{n(R)} \frac{2\mu_{kn(R)}}{R-1} \int_1^R \frac{da_k}{a_k^j} = \frac{2}{R-1} \int_1^R \frac{dx}{x^j}.$$

Hence,

$$\mathbb{E}(p_{1n(R)}) = \frac{2 \log R}{R-1}$$

and

$$\mathbb{E}(p_{jn(R)}) = \frac{2}{R-1} \frac{R^{j-1}-1}{(j-1)R^{j-1}}, \quad j > 1.$$

So, for all $j \geq 1$,

$$\lim_{R \to \infty} \mathbb{E}(p_{jn(R)}) = 0,$$

and the convergence is uniform with respect to $j \geq 1$. This implies that $p_{n(R)}$ converge almost surely to $p(z) \equiv 1$ as $R \to \infty$, and hence $f_{n(R)}$ converge almost surely to $f(z) = z$, $z \in \mathbb{D}$, as $R \to \infty$, which completes the proof of Theorem 3. □

Note that axiom **[UP1]**, as well as Theorem 3 and its proof, do not require that $n(R) \to \infty$ when $R \to \infty$.

Evidently, requirement **[UP2]** about the uniform distribution of points $\{a_k\}$ in $[1,R]$ can be replaced by other suitable continuous distributions, e.g., normal, exponential, geometric stable and so on. Discrete distributions are possible as well. Certainly, the conclusion of Theorem 3 can be changed under different assumptions.

For example, let $\varphi(a_k)$ be a probability density function with parameter $\lambda > 0$ for the exponential distribution located on $[1,R]$, $R > 1$,

$$\varphi(x) = \frac{\lambda e^{-\lambda(x-1)}}{1 - e^{-\lambda(R-1)}}, \quad x \in [1,R],$$

and $\varphi(x) = 0$ outside $[1, R]$.

Following the proof of Theorem 3 in this case, we come to the formula

$$\mathbb{E}(p_{jn(R)}) = 2\int\limits_1^R \frac{\varphi(x)}{x^j}dx = \frac{2\lambda}{1 - e^{-\lambda(R-1)}}\int\limits_1^R \frac{e^{-\lambda(x-1)}}{x^j}dx, \quad j \geq 1.$$

Send R to infinity and obtain

$$\lim_{R\to\infty} \mathbb{E}(p_{jn(R)}) = \frac{2}{\lambda}\int\limits_1^\infty \frac{e^{-\lambda(x-1)}}{x^j}dx > 0, \quad j \geq 1.$$

In such a case, if $\{a_k\}_{1\leq k\leq n(R)}$ has a limit configuration $\{a_n\}_{n=1}^\infty$ as $R \to \infty$, then the almost surely limit f is a starlike function in $\mathbb{D}$, and $p(z) = zf'(z)/f(z)$ is meromorphic with simple poles at $\{a_n\}_{n=1}^\infty$.

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