Value regions of univalent functions within different classes [1]

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Around the turn of the last century, the question of where an analytic function defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ or the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ can map points $z_0 \in \mathbb{D}$ or $z_0 \in \mathbb{H}$, respectively, was considered for different classes. Recently, numerous publications revived the interest to such problems, see, e.g., [2], [4-9].

The notable progress goes back to an idea by Loewner to express schlicht functions as solutions to a differential equation. Through Loewner's equation, it is possible to interpret an optimization problem for classes of univalent functions as the problem of finding a control that steers the trajectory of a dynamical system to the boundary of its reachable set. The most part of recent results in the general question has been obtained with the help of optimization technique. Powerful tools from the theory of optimal control can be applied to tackle the value region problem expressed via the Loewner equation.

Denote $\mathcal{H}(T)$, $T > 0$, the set of conformal maps from $\mathbb{H} \setminus K(T)$, with arbitrary hulls $K(T)$ of half-plane capacity T, onto $\mathbb{H}$, normalized hydrodynamically as

$$f_K(z) = z + \frac{2T}{z} + O\left(\frac{1}{|z|^2}\right), \quad \mathbb{H} \ni z \to \infty.$$

Roth and Schleissinger [9] found the set $\{f(z_0)\}$, $z_0 \in \mathbb{H}$, for the class $\cup_{T>0}\mathcal{H}(T)$. This research was continued in [7] for the class $\mathcal{H}(T)$ with fixed T. Without loss of generality, assume that $z_0 = i$ and consider the extremal problem to describe the value region

$$D(T) = \{f(i) : f \in \mathcal{H}(T),\ i \notin K(T)\}.$$

To formulate the result for $0 \le T \le \frac{1}{4}$, denote by $C_0(\varphi, T) > 0$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, $0 \le T \le \frac{1}{4}$, the unique root of the equation

$$2\cos^2\varphi \log(1 - \sin\varphi) + (1 - \sin\varphi)^2 = 2\cos^2\varphi \log C + C^2(1 - 4T).$$

For a fixed $T \in (0, \frac{1}{4}]$, this equation has a unique solution $C = C_0(\varphi, T)$ depending on φ.

Theorem 1. *The domain $D(T)$, $0 < T \le \frac{1}{4}$, is bounded by two curves l_1 and l_2 connecting the points i and $i\sqrt{1-4T}$. The curve l_1 in the complex (u, v)-plane is parameterized by the equations*

$$u(T) = \frac{C_0^2(\varphi, T)(4T - 1) + (1 - \sin\varphi)^2}{2C_0(\varphi, T)\cos\varphi}, \quad v(T) = \frac{1 - \sin\varphi}{C_0(\varphi, T)}, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}.$$

The curve l_2 is symmetric to l_1 with respect to the imaginary axis.

Denote by $\varphi_0(T) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $T > \frac{1}{4}$, the unique solution of the equation

$$\log\frac{1 - \sin\varphi}{1 + \sin\varphi} + \frac{1 - \sin\varphi}{1 + \sin\varphi} + 1 = \log\frac{1}{4T - 1}.$$

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For $T > \frac{1}{4}$ and $\varphi \in [\varphi_0(T), \frac{\pi}{2}]$, denote by $C_0(\varphi, T) > 0$ the minimal root of this equation and by $C_{00}(\varphi, T) > 0$ its maximal root.

Theorem 2. *The domain $D(T)$, $T > \frac{1}{4}$, is bounded by two curves $l_1 = l_{11} \cup l_{12}$ and $l_2 = l_{21} \cup l_{22}$ having a mutual endpoint $i \in l_{11} \cap l_{21}$. The curve l_{11} in the complex (u, v)-plane is parameterized by the equations*

$$u(T) = \frac{C_0^2(\varphi, T)(4T - 1) + (1 - \sin\varphi)^2}{2C_0(\varphi, T)\cos\varphi}, \quad v(T) = \frac{1 - \sin\varphi}{C_0(\varphi, T)}, \quad \varphi_0(T) \le \varphi \le \frac{\pi}{2}.$$

The curve l_{12} is parameterized by the equations

$$u(T) = \frac{C_{00}^2(\varphi, T)(4T - 1) + (1 - \sin\varphi)^2}{2C_{00}(\varphi, T)\cos\varphi}, \quad v(T) = \frac{1 - \sin\varphi}{C_{00}(\varphi, T)}, \quad \varphi_0(T) \le \varphi \le \frac{\pi}{2}.$$

The curve l_2 is symmetric to l_1 with respect to the imaginary axis.

The same approach was used to draw similar value regions for inverse functions $f^{-1} : \mathbb{H} \to \mathbb{H} \setminus K(T)$.

We observe many value region problems solved in [4-6]. For instance, the authors determine value sets $\{f(z_0)\}$, $z_0 \in \mathbb{D}$, and $\{f^{-1}(w_0)\}$ for the class

$$\mathcal{I} = \{f \in \mathcal{H} : f(-\overline{z}) = -\overline{f(z)}, z \in \mathbb{H}\},$$

where $\mathcal{H}$ is the class of univalent self-mappings of $\mathbb{H}$ with the hydrodynamic normalization. Going to self-maps of $\mathbb{D}$, they prove a result which is equivalent to the classical solution by Goryainov and Gutlyanski [1] in the class $S(M)$, $M > 1$, $S(\infty) = S$, of univalent functions f in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1$, and $|f(z)| < M$ in $\mathbb{D}$. The same is done for typically real functions in $\mathbb{D}$ and some other classes.

Remind a theorem by Fedorov [3] which gives a value region $\{f(z_0)\}$, $z_0 \in \mathbb{D}$, over the class S_R of functions $f \in S$ with real values $f^{(n)}(0)$, $n \ge 2$. Obviously, an answer is easy if z_0 is real. From the other hand, it is strongly nontrivial when $\operatorname{Im} z_0 \ne 0$. Fedorov's result is extended in [8] to the class $S_R(M) = S_R \cap S(M)$. Usually, a subclass is organized more properly than a whole class of functions. However, it is not the case when we pass from S to $S(M)$ or from S_R to $S_R(M)$. Fedorov completely solved the problem by simultaneously considering two moduli problems for pairs of homotopic classes of curves. In $S_R(M)$, the problem is formulated as a reachable set problem for the Hamilton system of controllable differential equations in the frames of the Loewner theory. A family of Cauchy problems is substituted for the family of boundary value problems. The free parameter in the initial data serves as a parameter for the boundary curve of the value region. We do not write down a theorem proved in [8] since it requires too large volume.

Finally, let us concern with a class of holomorphic injective self-maps $f : \mathbb{D} \to \mathbb{D}$ having boundary fixed points, the class actively investigated during last decades by Goryainov, among others. For the dynamics of a self-map $f : \mathbb{D} \to \mathbb{D}$, a crucial role is played by the points $\sigma \in \partial\mathbb{D}$ for which $f(\sigma) := \angle\lim_{z\to\sigma} f(z) = \sigma$ and the angular derivative $f'(\sigma)$ is finite. Such points σ are called boundary regular fixed points. In particular, a classical result due to Wolff and Denjoy asserts that if $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$ has no fixed points in $\mathbb{D}$, then it possesses the so-called Denjoy-Wolff point, i.e., a unique boundary fixed point τ such that $f'(\tau) \le 1$.

Consider univalent self-maps $f : \mathbb{D} \to \mathbb{D}$ with a given boundary regular fixed point $\sigma \in \partial\mathbb{D}$ and the Denjoy-Wolff point $\tau \in \partial\mathbb{D} \setminus \{\sigma\}$. Using automorphisms of $\mathbb{D}$, we may suppose that $\tau = 1$ and $\sigma = -1$. We mean to determine a sharp value region of $f \mapsto f(z_0)$, $z_0 \in \mathbb{D}$, for all such self-maps of $\mathbb{D}$ with $f'(-1)$ fixed. Fix $z_0 \in \mathbb{D}$, $T > 0$ and let $\zeta_0 = x_1^0 + ix_2^0 := l(z_0)$, where

$$l : \mathbb{D} \to \mathbb{S}; \quad z \mapsto \log\frac{1+z}{1-z}$$

is a conformal map of $\mathbb{D}$ onto the strip $\mathbb{S} := \{\zeta\colon -\pi/2 < \operatorname{Im}\zeta < \pi/2\}$. Define

$$a_\pm(T) := e^{-T/2}\sin x_2^0 \pm (1-e^{-T/2}), \quad R(a,T) := \log\frac{1-a}{1-a_+(T)}\ \log\frac{1+a}{1+a_-(T)},$$

$$V(\zeta_0, T) := \Big\{x_1+ix_2 \in \mathbb{S}\colon a_-(T) \le \sin x_2 \le a_+(T),\ \big|x_1 - x_1^0 - \tfrac{T}{2}\big| \le \sqrt{R(\sin x_2, T)}\Big\}.$$

Theorem 3. *Let $f \in Hol(\mathbb{D},\mathbb{D}) \setminus \{id_{\mathbb{D}}\}$ and $T > 0$. Suppose that:*
(i) f is univalent in $\mathbb{D}$;
(ii) the Denjoy-Wolff point of f is $\tau = 1$;
(iii) $\sigma = -1$ is a boundary regular fixed point of f and $f'(-1) = e^T$.
Then

$$f(z_0) \in \mathcal{V}(z_0,T) := l^{-1}\big(V(l(z_0),T)\big) \setminus \{z_0\}$$

for any $z_0 \in \mathbb{D}$. This result is sharp, i.e., for any $w_0 \in \mathcal{V}(z_0,T)$ there exists $f \in Hol(\mathbb{D},\mathbb{D}) \setminus \{id_{\mathbb{D}}\}$, satisfying (i) – (iii) and such that $f(z_0) = w_0$.

Characterize functions f corresponding to boundary points of $\mathcal{V}(z_0,T)$. The role of the Koebe function $f_0(z) = z(1-z)^{-2}$in S is played by the Pick function $p_M(z) := f_0^{-1}(f_0(z)/M)$, $M > 1$.

Theorem 4. *For any $w_0 \in \partial V(z_0,T) \setminus \{z_0\}$, there exists a unique $f = f_{w_0}$ satisfying conditions (i)-(iii) in Theorem 3 and such that $f_{w_0}(z_0) = w_0$. If $w_0 = l^{-1}(\zeta_0+T)$, then f_{w_0} is a hyperbolic automorphism of $\mathbb{D}$, namely, $f_{w_0}(z) = l^{-1}(l(z)+T)$. Otherwise, f_{w_0} is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$ minus a slit along an analytic Jordan curve γ orthogonal to $\partial\mathbb{D}$, with $f'_{w_0}(1) = 1$. Moreover, $f_{w_0} = h_1 \circ p_M \circ h_2$ for some $h_1, h_2 \in Aut(\mathbb{D})$ and $M > 1$ if and only if $w_0 = l^{-1}(x_1^0 + T/2 + i\arcsin a_\pm(T))$.*

Note that z_0 is a boundary point of the value region $\mathcal{V}(z_0,T)$, but does not belong to $\mathcal{V}(z_0,T)$. This point z_0 would be included, and this would be the only modification of the value region, if we replace the equality $f'(-1) = e^T$ in condition (iii) of Theorem 3 by the inequality $f'(-1) \le e^T$ and remove the requirement $f \ne \mathrm{id}_{\mathbb{D}}$ assuming as a convention that $\mathrm{id}_{\mathbb{D}}$ satisfies (ii). Note also that, under the conditions of Theorem 3 modified in this way, $f(z_0) = z_0$ if and only if $f = \mathrm{id}_{\mathbb{D}}$.

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