Spectral Analysis of Random Processes

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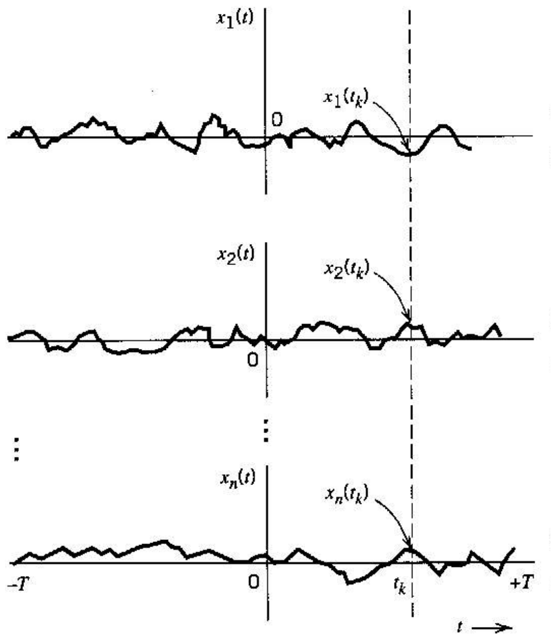


Generally, all properties of a random process should be defined by **averaging over the ensemble of realizations.**

Generally, all **realizations** of the same random process **are different**.

Intuitively, many random processes possess some **internal timescales**, that are visible from realizations.

One wants to analyse **frequency content** of a random process in a way similar to the one for deterministic functions.

There are two general ways to introduce a **spectral characteristic of a random process**.

NB: Although the process is random, its ***autocorrelation*** function is a ***deterministic*** function!

Power Spectral Density: Wiener-Khinchine Theorem

If *X(t)* is a wide-sense stationary random process with autocorrelation function $K_{XX}(\tau)$, its **power spectral density** $S(\omega)$ can be introduced as a Fourier Transform of $K_{XX}(\tau)$:

$$S(\omega) = \Im(K_{XX}(\tau)) = \int_{-\infty}^{\infty} K_{XX}(\tau) e^{-j\omega\tau} \, d\tau$$

Then autocorrelation function $K_{XX}(\tau)$ is then an inverse FT of power spectral density $S(\omega)$:

$$K_{XX}(\tau) = \Im^{-1}(S(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} \, d\omega$$

Wiener-Khinchine Theorem: Power spectral density and autocorrelation function of a wide-sense stationary random process are related via Fourier Transform.

Power of a WSS Process

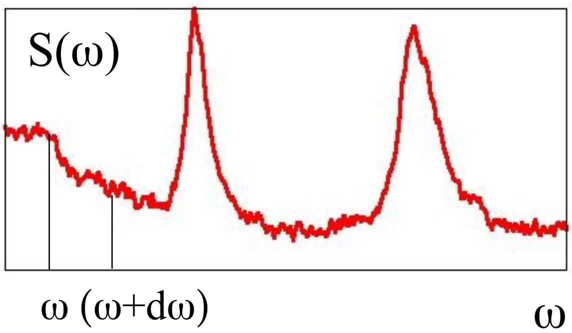
Consider FT of autocorrelation: $K_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} \, d\omega$

If τ=0, then: $K_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{0} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = P$

The integral over the power spectral density is a **power *P*** of a Wide-Sense Stationary process. i.e. **power *P*** is equal to the value of autocorrelation function $K_{XX}(\tau)$ at τ=0.

$$K_{XX}(0) = \overline{X(t)X(t+\tau)}\big|_{\tau=0} = \overline{X^2} = P$$

$$P = K_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega$$



Meaning of Power Spectral Density S(ω) :
The value of S(ω) is the power concentrated in a frequency interval [ω; ω+dω].

Calculation of Power

Calculation from mean and variance

If one knows ***mean*** and ***variance*** of the random process, the power P can be calculated using the following formulae:

$$P = \overline{X^2}$$

$$\sigma_X^2 = \overline{X^2} - \left(\overline{X}\right)^2$$

$$P = \sigma_X^2 + \left(\overline{X}\right)^2$$

Calculation from power spectral density

If one knows ***power spectral density*** *S* of the process in terms of radial frequency ω, or of plain frequency *f*, the power *P* is an integral over *S*:

$$P = K_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega = \int_{-\infty}^{\infty} S(f)df$$

NB: The factor (1/2π) vanishes if the spectrum is known in terms of plain frequency *f*.

Properties of Power Spectral Density of Wide-Sense Stationary Process

1) $S(\omega) \geq 0$ PSD *S*(ω) is ***positive***, because the meaning of S(ω) is the power inside the frequency interval [ω,ω+Δω].

2) *S*(ω) is ***real***, because power is real.

3) *S*(ω) is an ***even*** function of frequency ω. Let us ***prove*** this.

Consider $S(-\omega) = S(\omega)$ Property of ACF : $K_{XX}(-\tau) = K_{XX}(\tau) \quad \Rightarrow$

$$S(\omega) = \int_{-\infty}^{\infty} K_{XX}(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty} K_{XX}(\tau)\left(\cos(\omega\tau) - j\sin(\omega\tau)\right)d\tau =$$

$$= \int_{-\infty}^{\infty} K_{XX}(\tau)\cos(\omega\tau)d\tau - j\underbrace{\int_{-\infty}^{\infty} K_{XX}(\tau)\sin(\omega\tau)d\tau}_{=0} = 2\int_{0}^{\infty} K_{XX}(\tau)\cos(\omega\tau)d\tau;$$

$$S(-\omega) = \int_{-\infty}^{\infty} K_{XX}(\tau)e^{j\omega\tau}\,d\tau = \int_{-\infty}^{\infty} K_{XX}(\tau)\left(\cos(\omega\tau) + j\sin(\omega\tau)\right)d\tau =$$

$$= \int_{-\infty}^{\infty} K_{XX}(\tau)\cos(\omega\tau)d\tau = S(\omega).$$

4) The ***wider the spectrum***, the ***narrower the autocorrelation*** function is, and vice versa.

Mini-test

1. The mean value of a wide sense stationary random process is 4, and the variance is 7. What is the power of the process?

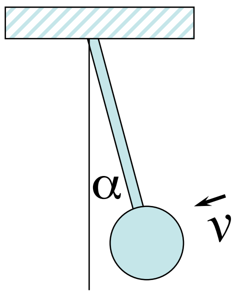
2. Which of the following functions could be valid power spectrum density of a stationary process -

a) $S(\omega) = j\dfrac{\omega^2 + 4}{\omega^4 + 8}$ b) $S(\omega) = \dfrac{\omega^2 + 4}{\omega^4 + 8}$

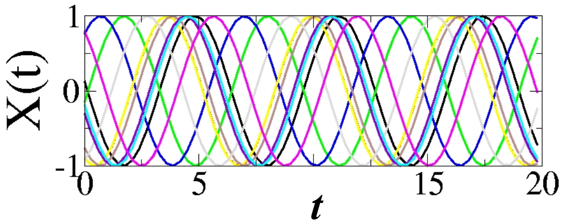
c) $S(\omega) = \dfrac{\omega^2 + 4}{\omega^4 - 8}$ d) $S(\omega) = \dfrac{\omega^3 + 4}{\omega^6 - 8}$

Example: Cosine with Random Phase (PS. 2, Q. 1)

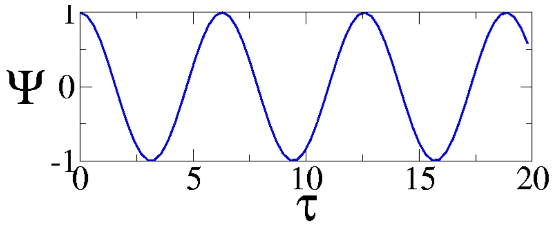
Given: $X(t)=A\cos(\omega_0 t+\varphi)=f(\varphi,t)$, where A,ω_0 are constants, φ is a random variable uniformly distributed in the interval $[-\pi,\pi]$.



An example of a real system where such a process can be realized is a pendulum without friction whose amplitude of oscillations is small. Initial push with velocity v together with the initial angle α define the phase φ.



Different realizations of $X(t)$ are shown. The process $X(t)$ is wide sense stationary.

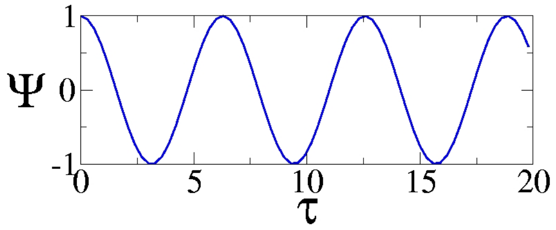


Covariance of $X(t)$ found earlier reads:

$$\Psi_{XX}(t,\tau) = \frac{A^2}{2}\cos(\omega_0\tau) = \Psi_{XX}(\tau)$$

Example: Cosine with Random Phase (PS. 2, Q. 1) (continued…)

$$\Psi_{XX}(t,\tau) = \frac{A^2}{2}\cos(\omega_0\tau) = \Psi_{XX}(\tau)$$



A natural time scale present in the process is frequency ω_0.
Autocorrelation (ACF) is equal to covariance.
ACF oscillates with frequency ω_0.
Power spectral density (PSD) of this process is

$$S(\omega) = \frac{A_0^2\pi}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$$

I.e. PSD contains a component at the frequency ω_0.

Power Spectral Density of a Random Process: from ACF and Directly from Realizations

Power Spectral Density from experimental data

Problems arising when estimating from a time series:

1. In order to introduce power spectral density by means of Wiener-Khinchine Theorem, one first has to estimate ***autocorrelation*** function of the process. However, with experimental data this can be a separate problem requiring a ***lot of calculation.***

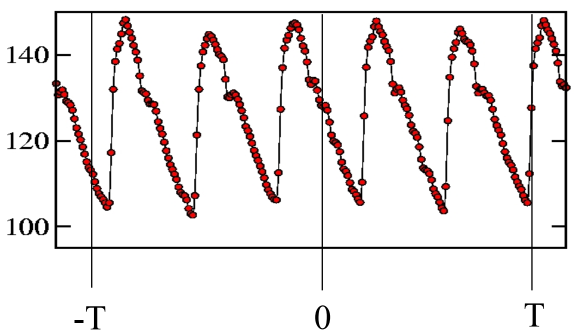
2. Whatever is measured during experiment, will have ***finite duration***. Therefore, any estimate will be made with a certain ***mistake***.

Question: Can power spectral density for a random process be introduced straightforwardly for the experimental signal?

Power Spectral Density from Realizations of a Random Process

We have introduced FT for a deterministic function of time.
Note, that FT exists only for functions that are absolutely integrable

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$



How can a function be absolutely integrable? If it is nonzero on a finite time interval only, or if it asymptotically tends to zero as time t tends to plus or minus infinity.

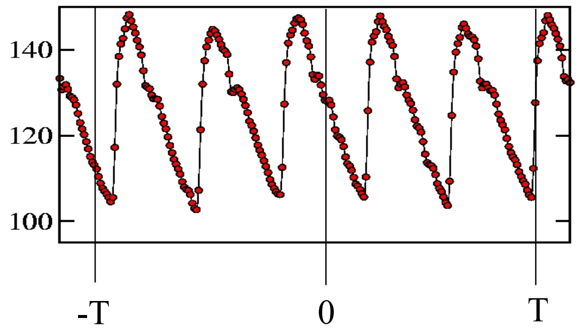
Some exceptions are mentioned in Lecture 12.

Now, since we study **random processes**, we want to be able to estimate the spectral content of it. We can introduce FT of a realization of random process by analogy to the FT of a deterministic function of time. However, most random processes of our interest are NOT absolutely integrable! Therefore, their FT most probably does not exist. **Can we do anything about it?**

Power Spectral Density from Realizations of a Random Process (cont…)

Consider random process $X(t)$ with finite variance, and its particular realization $x(t)$.
Define $x_T(t)$ to be a portion of $x(t)$ inside the time interval [$-T$; T]:

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & \text{elsewhere} \end{cases}$$



Because T is finite, the integral over $|x_T(t)|$ will be finite:

$$\int_{-\infty}^{\infty} |x_T(t)|\,dt = \int_{-T}^{T} |x_T(t)|\,dt < \infty$$

Therefore, $x_T(t)$ will have FT, which we denote $F_T(\omega)$:

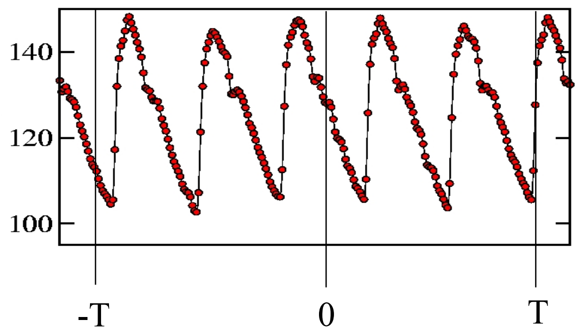
$$F_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t}\,dt = \int_{-T}^{T} x(t) e^{-j\omega t}\,dt$$

Energy and power of the process

Consider energy W_T and power P_T of $x(t)$ inside a finite time interval [$-T$; T]

$$W_T = \int_{-T}^{T} |x(t)|^2\,dt, \qquad P_T = \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$$

Note: Power is energy per time unit.
Then: Let T tend to infinity.
Condition: Let for almost all realizations $x(t)$ of random process $X(t)$ the ***power*** P of the whole process be ***finite***, i.e.



$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$$

$$0 < P < \infty$$

Processes with ***finite power*** can be analysed by means of Fourier Transform.

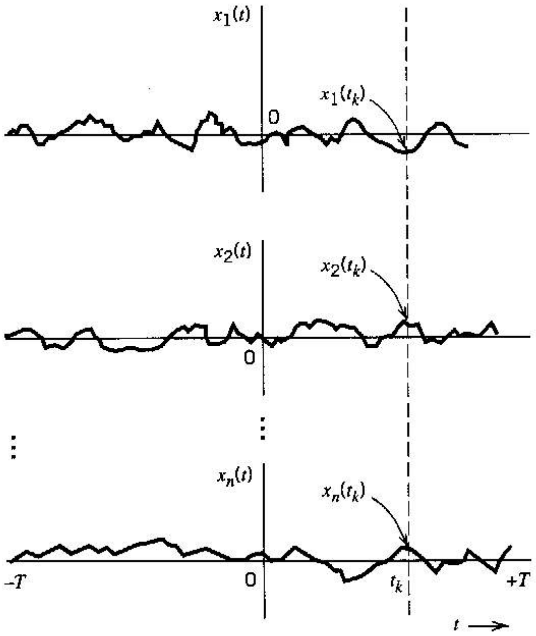
Remark: The ***energy*** of the infinitely long process can be ***infinite***, but its ***power*** can be ***finite***!

Random Spectrum

Let: *X(t)* be a wide-sense stationary random process.

For a particular realization of a random process of duration *2T* one can introduce Fourier Transform, or Fourier Spectrum (another term) as follows:

$$F_T(\omega) = \int_{-T}^{T} x(t) e^{-j\omega t}\, dt$$



Note: For ***different realizations*** of the same random process ***Fourier Spectrum*** will be ***different***. In fact, it will randomly change from one realization to another. Therefore, spectrum $F_T(\omega)$ of *x(t)* can be called a **random spectrum.**

Summary: Power Spectral Density from Realizations of a Random Process

For a random process *X(t)* a **power spectral density** $S(\omega)$ can be ***introduced*** as follows:

1) Calculate the ***Fourier Transform*** $F_T(\omega)$ for a realization $x(t)$ of duration *2T* as

$$F_T(\omega) = \int_{-T}^{T} x(t) e^{-j\omega t}\, dt$$

and $|F_T(\omega)|^2$ as: $$\left|F_T(\omega)\right|^2 = F_T(\omega) F_T^*(\omega)$$

2) ***Average*** $|F_T(\omega)|^2$ over the ensemble of realizations of the process to obtain $\overline{\left|F_T(\omega)\right|^2}$

3) Introduce ***power*** $P_T(\omega)$ witihin a finite time interval [-*T*;*T*]: $\frac{1}{2T}\overline{\left|F_T(\omega)\right|^2} = P_T(\omega)$

4) Take the ***limit*** of $P_T(\omega)$ as T tends to infinity to obtain power spectral density $S(\omega)$:

This is the power of the process in the frequency band [ω; ω+dω], i.e.

$$\boxed{S(\omega) = \lim_{T\to\infty} P_T(\omega) = \lim_{T\to\infty} \frac{1}{2T}\overline{\left|F(\omega)\right|^2}}$$

PSD of cosine with random phase
(PS 8, Q. 3)

Given: random process $X(t)=A_0\cos(\omega_0 t+\varphi)=f(\varphi,t)$, where A_0, ω_0 are constants, $\varphi$is a random variable uniformly distributed in the interval $[-\pi,\pi]$.

Find: power spectral density $S(\omega)$ of $X(t)$ by two methods: using Wiener-Khinchine theorem and from periodograms. Sketch power spectral density.

Hint: Autocorrelation function of $X(t)$ was found within PS 2, Q. 1 to be:

$$K_{XX}(t,\tau)=\frac{A_0^2}{2}\cos\omega_0\tau$$

PSD of cosine with random phase
from Wiener-Khintchine theorem (PS 8, Q. 3)

Solution: 1) Find power spectral density $S(\omega)$ using Wiener-Khinchine theorem, i.e

$$\boxed{\cos(\omega_0 t)=\frac{\exp(j\omega_0 t)+\exp(-j\omega_0 t)}{2}}$$

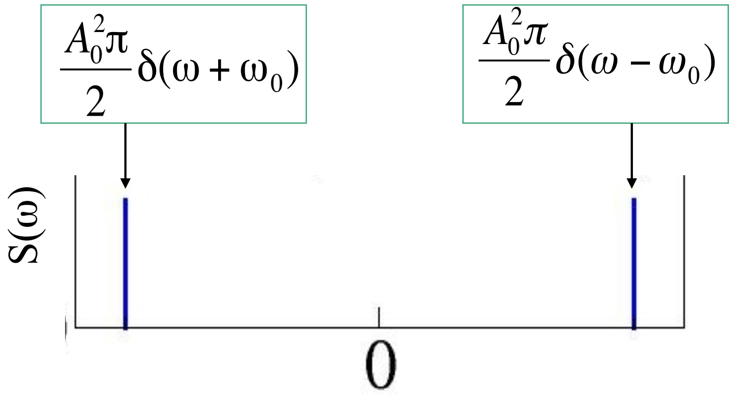
$$S(\omega)=\int_{-\infty}^{\infty}K_{XX}(\tau)e^{-j\omega\tau}\,d\tau=\frac{A_0^2}{2}\int_{-\infty}^{\infty}\cos(\omega_0\tau)e^{-j\omega\tau}\,d\tau$$

$$=\frac{A_0^2}{2\cdot 2}\left[\int_{-\infty}^{\infty}e^{j\omega_0\tau}e^{-j\omega\tau}\,d\tau+\int_{-\infty}^{\infty}e^{-j\omega_0\tau}e^{-j\omega\tau}\,d\tau\right] \qquad \boxed{\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\,d\omega}$$

$$=\frac{A_0^2}{4}\left[2\pi\delta(\omega-\omega_0)+2\pi\delta(\omega+\omega_0)\right]=\frac{A_0^2\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

PSD of cosine with random phase from Wiener-Khintchine theorem (PS 8, Q. 3)

$$\boxed{S(\omega) = \frac{A_0^2\pi}{2}\delta(\omega-\omega_0) + \frac{A_0^2\pi}{2}\delta(\omega+\omega_0)}$$



This is the ***power spectral density*** of cosine with random phase.

PSD of cosine with random phase from realizations

2) Now introduce power spectral density ***from realizations*** of random process $X(t)=A_0\cos(\omega_0 t+\varphi)$, where φ is a random variable uniformly distributed between $-\pi$ and $+\pi$.

2.1 First, we need to find the **"finite time" Fourier Transform** of a single realization, i.e. $F_T(\omega)$

$$F_T(\omega) = \int_{-T}^{T} A_0\cos(\omega_0 t+\varphi)e^{-j\omega t}\,dt = \frac{A_0}{2}\int_{-T}^{T} e^{j\omega_0 t+j\varphi}e^{-j\omega t}\,dt + \frac{A_0}{2}\int_{-T}^{T} e^{-j\omega_0 t-j\varphi}e^{-j\omega t}\,dt$$

$$= \frac{A_0}{2}\left[e^{j\varphi}\int_{-T}^{T} e^{j(\omega_0-\omega)t}\,dt + e^{-j\varphi}\int_{-T}^{T} e^{-j(\omega_0+\omega)t}\,dt\right]$$

$$= \frac{A_0}{2}\left[e^{j\varphi}\frac{1}{j(\omega_0-\omega)}e^{j(\omega_0-\omega)t}\Bigg|_{-T}^{T} + e^{-j\varphi}\frac{1}{-j(\omega_0+\omega)}e^{-j(\omega_0+\omega)t}\Bigg|_{-T}^{T}\right]$$

PSD of cosine with random phase from realizations (cont…)

$$F_T(\omega)=\frac{A_0}{2}\left[e^{j\varphi}\left(e^{j(\omega_0-\omega)T}-e^{-j(\omega_0-\omega)T}\right)\frac{2j}{2j\cdot j(\omega_0-\omega)}+\right.$$

$$\left.+e^{-j\varphi}\left(e^{-j(\omega_0+\omega)T}-e^{j(\omega_0+\omega)T}\right)\frac{2j}{2j\cdot(-j)(\omega_0+\omega)}\right]$$

Take into account that sine can be represented through exponents:

$$\boxed{\sin(\omega_0 t)=\frac{\exp(j\omega_0 t)-\exp(-j\omega_0 t)}{2j}}$$

$$=\frac{A_0}{2}\left[e^{j\varphi}\sin[(\omega_0-\omega)T]\frac{2}{(\omega_0-\omega)}+e^{-j\varphi}\sin[(\omega_0+\omega)T]\frac{2}{(\omega_0+\omega)}\right]=$$

$$=A_0Te^{j\varphi}\frac{\sin[(\omega_0-\omega)T]}{(\omega_0-\omega)T}+A_0Te^{-j\varphi}\frac{\sin[(\omega_0+\omega)T]}{(\omega_0+\omega)T}$$

PSD of cosine with random phase from realizations (cont…)

Now, find the **square of the magnitude** of $F_T(\omega)$: $|F_T(\omega)|^2=F_T(\omega)F_T^*(\omega)$

Denote:

$$F_T(\omega)=c,\quad A_0T\frac{\sin[(\omega_0-\omega)T]}{(\omega_0-\omega)T}=a,\quad A_0T\frac{\sin[(\omega_0+\omega)T]}{(\omega_0+\omega)T}=b$$

Calculate cc^*:

$$c=ae^{j\varphi}+be^{-j\varphi}=a\cos\varphi+ja\sin\varphi+b\cos\varphi-jb\sin\varphi$$

$$c^*=a\cos\varphi-ja\sin\varphi+b\cos\varphi+jb\sin\varphi=ae^{-j\varphi}+be^{j\varphi}$$

$$cc^*=|c|^2=\left(ae^{j\varphi}+be^{-j\varphi}\right)\left(ae^{-j\varphi}+be^{j\varphi}\right)=a^2+b^2+abe^{2j\varphi}+abe^{-2j\varphi}$$

PSD of cosine with random phase from realizations (cont…)

2.2 We now need to find **mean of |c|²** averaged over the ensemble of realizations of this random process, i.e. in fact over φ. *a, b* and their combinations do not depend on φ, and thus do not participate in averaging over φ: they are treated as constants.

$$\overline{|c|^2} = \overline{a^2} + \overline{b^2} + \overline{abe^{2j\varphi}} + \overline{abe^{-2j\varphi}} = a^2 + b^2 + ab\overline{e^{2j\varphi}} + ab\overline{e^{-2j\varphi}}$$

$$\overline{e^{2j\varphi}} = \int_{-\infty}^{\infty} e^{2j\varphi} p_1(\varphi)d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2j\varphi}d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(2\varphi)d\varphi + j\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(2\varphi)d\varphi = 0$$

$$\overline{e^{-2j\varphi}} = 0$$

Thus, **average** over the ensemble of realizations of $|F_T(\omega)|^2$ is

$$\overline{|F_T(\omega)|^2} = \left(A_0T\right)^2\left(\frac{\sin\left[(\omega_0-\omega)T\right]}{(\omega_0-\omega)T}\right)^2 + \left(A_0T\right)^2\left(\frac{\sin\left[(\omega_0+\omega)T\right]}{(\omega_0+\omega)T}\right)^2$$

PSD of cosine with random phase from realizations (cont…)

2.3 Now we have to find the following **limit** as observation time ***T* tends to infinity**

$$\lim_{T\to\infty}\frac{1}{2T}\overline{|F_T(\omega)|^2} = \lim_{T\to\infty}\frac{1}{2T}\left\{\left(A_0T\right)^2\left(\frac{\sin\left[(\omega_0-\omega)T\right]}{(\omega_0-\omega)T}\right)^2 + \left(A_0T\right)^2\left(\frac{\sin\left[(\omega_0+\omega)T\right]}{(\omega_0+\omega)T}\right)^2\right\}$$

$$= \lim_{T\to\infty}\left\{\frac{A_0^2T}{2}\left(\frac{\sin\left[(\omega_0-\omega)T\right]}{(\omega_0-\omega)T}\right)^2 + \frac{A_0^2T}{2}\left(\frac{\sin\left[(\omega_0+\omega)T\right]}{(\omega_0+\omega)T}\right)^2\right\}$$

$$= \boxed{\frac{A_0^2}{2}\pi\delta(\omega_0-\omega) + \frac{A_0^2}{2}\pi\delta(\omega_0+\omega) = S(\omega)}$$

$$\lim_{T\to\infty} T\left[\frac{\sin(\alpha T)}{\alpha T}\right]^2 = \pi\delta(\alpha)$$

One can prove the validity of the above limit, but here we will take it for granted.

Conclusion:
Thus, power spectral densities introduced by two ways for the same process cosine of random phase, ***coincide***!

Why Did we Have to Learn the Previous Material?

In experimental situation you are going to consider single realizations of random processes.

Question: If power spectral density is introduced from an ensemble of realizations, or from autocorrelation function of the process, what is the use of it if we do not have ensemble of realizations?

Answer: There is a class of random processes that are **ergodic**. For them all or some of their important statistical characteristics can be estimated from **averaging over time**. We thus do not need an ensemble of realizations to estimate autocorrelation function of an ergodic process, only one realization will be sufficient.

If ***one realization*** contains full information about the autocorrelation function (ACF), then it must contain ***full information about*** the Fourier Transform of ACF. The latter is ***power spectral density***!