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Смешанная задача с косой производной для телеграфного уравнения с нелинейным потенциалом[1]

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Для телеграфного уравнения с нелинейным потенциалом, заданного в первом квадранте, изучается смешанная задача, в которой на пространственной полуоси задаются условия Коши, а на временной полуоси задаётся условие, которое содержит производную по направлению (косую производную), зависящему от времени. Рассмотрены вопросы существования и единственности глобального классического решения.

Ключевые слова: нелинейное волновое уравнение, полулинейное уравнение, метод характеристик, классическое решение, смешанная задача, условия согласования.

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Mixed problem with a directional derivative for the telegraph equation with a nonlinear potential[1]

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For the telegraph equation with a nonlinear potential given in the first quadrant, we study a mixed problem in which the Cauchy conditions are specified on the spatial half-line and the condition, which contains a directional (oblique) derivative with time-dependent direction, is specified on the time half-line. The existence and uniqueness of a global classical solution are considered.

Keywords: nonlinear wave equation, semilinear equation, method of characteristics, classical solution, mixed problem, matching conditions.

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Statement of the problem

In the closure $\overline{Q}$ of the domain $Q = (0, \infty) \times (0, \infty)$ of two independent variables (t, x) for the nonlinear equation

$$\partial_t^2 u(t,x) - a^2 \partial_x^2 u(t,x) = f(t,x,u(t,x)), \tag{1}$$

where $a > 0$, we consider the mixed problem with the initial conditions

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \quad x \in [0,\infty) \tag{2}$$

and one of the following boundary conditions:

$$\alpha(t)\partial_t u(t,0) + \beta(t)\partial_x u(t,0) + \gamma(t)u(t,0) = \mu(t), \quad t \in [0,\infty), \tag{3}$$

where $a\alpha(t) \neq \beta(t)$ for all $t \in [0,\infty)$, or

$$\frac{\beta(t)}{a}\partial_t u(t,0) + \beta(t)\partial_x u(t,0) + \gamma(t)u(t,0) = \mu(t), \quad t \in [0,\infty). \tag{4}$$

Main result

Theorem 1. *Let the conditions*

$$f \in C^1(\overline{Q} \times \mathbb{R}), \quad \varphi \in C^2([0,\infty)), \quad \psi \in C^1([0,\infty)), \quad \mu \in C^1([0,\infty)),$$
$$\alpha \in C^1([0,\infty)), \quad \beta \in C^1([0,\infty)), \quad \gamma \in C^1([0,\infty))$$

be satisfied, and let the function f satisfy a Lipschitz type condition with respect to the third variable; i.e., assume that there exists a function $k \in L^2_{\mathrm{loc}}(\overline{Q})$ such that $|f(t,x,z_1) - f(t,x,z_2)| \leqslant k(t,x)|z_1 - z_2|$. The mixed problem (1) – (3) has a unique solution u in the class $C^2(\overline{Q})$ if and only if conditions

$$\mu(0) = \alpha(0)\psi(0) + \beta(0)\varphi'(0) + \gamma(0)\varphi(0), \tag{5}$$

$$\mu'(0) = \alpha(0)\left(f(0,0,\varphi(0)) + a^2\varphi''(0)\right) + \psi(0)\alpha'(0) + \beta'(0)\varphi'(0) + \\ + \beta(0)\psi'(0) + \varphi(0)\gamma'(0) + \gamma(0)\psi(0) = \mu'(0) \tag{6}$$

are satisfied.

Theorem 2. *Let the conditions*

$$f \in C^1(\overline{Q} \times \mathbb{R}), \quad \varphi \in C^3([0,\infty)), \quad \psi \in C^2([0,\infty)),$$
$$\beta \in C^2([0,\infty)), \quad \gamma \in C^2([0,\infty)), \quad \mu \in C^2([0,\infty))$$

be satisfied, let the inequalities $\gamma(t) \neq 0$ and $\gamma'(t) \neq 0$ be fulfilled for all $t \in [0,\infty)$, and let the function f satisfy a Lipschitz type condition with respect to the third variable; i.e., assume that there exists a function $k \in C(\overline{Q})$ such that $|f(t,x,z_1) - f(t,x,z_2)| \leqslant k(t,x)|z_1 - z_2|$. The mixed problem (1), (2) and (4) has a unique solution u in the class $C^2(\overline{Q})$ if and only if conditions

$$\mu(0) = \frac{\beta(0)\psi(0)}{a} + \beta(0)\varphi'(0) + \gamma(0)\varphi(0),$$

$$\mu'(0) = \frac{\beta(0)\left(f(0,0,\varphi(0)) + a^2\varphi''(0)\right)}{a} + \frac{\psi(0)\beta'(0)}{a} + \beta'(0)\varphi'(0) +$$
$$+ \beta(0)\psi'(0) + \varphi(0)\gamma'(0) + \gamma(0)\psi(0),$$
$$\mu''(0) = \beta(0)\left(\varphi'(0)\partial_u f(0,0,u=\varphi(0)) + \partial_x f(0,0,\varphi(0)) + a^2\varphi'''(0)\right) +$$
$$+ \frac{\beta(0)\left(\psi(0)\partial_u f(0,0,u=\varphi(0)) + \partial_t f(0,0,\varphi(0)) + a^2\psi''(0)\right)}{a} +$$
$$+ \frac{2\beta'(0)\left(f(0,0,\varphi(0)) + a^2\varphi''(0)\right)}{a} + \gamma(0)\left(f(0,0,\varphi(0)) + a^2\varphi''(0)\right) +$$
$$+ \frac{\psi(0)\beta''(0)}{a} + \beta''(0)\varphi'(0) + 2\beta'(0)\psi'(0) + \varphi(0)\gamma''(0) + 2\psi(0)\gamma'(0).$$

are satisfied.

The **proof** of Theorems 1 and 2 is carried out by the scheme set forth in [1–4] using the results of the papers [5, 6].

Remark 1. *Assume that the boundary data of the problem* (1), (2), (4) *satisfy the equality* $\beta \equiv 0$ *and the inequality* $\gamma(t) \neq 0$ *for all* $t \geqslant 0$. *Then the conditions specified in Theorem 2 can be weakened: 1) the smoothness conditions* $\varphi \in C^3([0,\infty))$ *and* $\psi \in C^2([0,\infty))$ *can be replaced with* $\varphi \in C^2([0,\infty))$ *and* $\psi \in C^1([0,\infty))$, *respectively; 2) the condition* $\gamma'(t) \neq 0$ *can be dropped.*

Proof. Under the conditions of this remark, the boundary condition (4) transforms into the Dirichlet condition

$$u(t,0) = \frac{\mu(t)}{\gamma(t)}, \quad t \in [0,\infty). \tag{7}$$

Thus, the mixed problem (1), (2), (4) transforms into the well-studied first mixed problem (1), (2) and (7) [1–3].

If the given functions of the problem (1) – (3) do not satisfy the homogeneous matching conditions (5) and (6), then the solution of this problem is reduced to solving the corresponding matching problem in which the matching conditions are given on the characteristic $x - at = 0$. The following condition can be taken for the matching condition:

$$[(u)^+ - (u)^-](t, x = at) = 0, \quad t \in [0,\infty). \tag{8}$$

Problem (1) – (3) with matching conditions on characteristics. Find a classical solution of Eq. (1) with the Cauchy conditions (2), the boundary condition (3), and the matching condition (8).

Here by $()^{\pm}$ we have denoted the limit values of the function calculated on different sides of the characteristic $x - at = 0$, i. e. $(u)^{\pm}(t, x = at) = \lim\limits_{\delta \to 0+} u(t, at \pm \delta)$.

Now the problem (1) – (3) can be stated using the matching conditions (8) as follows.

Problem (1) – (3) with matching conditions on characteristics. Find a classical solution of Eq. (1) with the Cauchy conditions (2), the boundary condition (3), and the matching condition (8).

The same can be said about the problem (1), (2) and (4), but instead of the homogeneous matching condition (8) we should take the following inhomogeneous condition:

$$[(u)^+ - (u)^-](t, x = at) = \frac{a\left(\beta(0)\varphi'(0) + \gamma(0)\varphi(0) - \mu(0)\right) + \beta(0)\psi(0)}{a\gamma(0)}, \quad t \in [0, \infty).$$

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