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THE CAUCHY SINGULAR INTEGRAL ON NON-SMOOTH CURVE

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Let Γ be a simple Jordan curve on the complex plane $\mathbb{C}$. The original definition of the Cauchy singular integral over this curve is

$$\mathcal{S}_\Gamma f(t) := \lim_{\epsilon\to 0}\frac{1}{\pi i}\int\limits_{\Gamma\setminus\{|\tau-t|\le\epsilon\}}\frac{f(\tau)d\tau}{\tau-t}, \quad t\in\Gamma. \tag{1}$$

This integral operator has a lot of applications. In particular, it is of importance for theory of boundary value problems for analytic functions, for aero and hydrodynamics, and for theory of elasticity, see [1–3]. There exists a great body of publications on this subject. Here we restrict our references by the classical monograph [4] and recent survey [5].

It is well known [1–3], that $\mathcal{S}_\Gamma f$ exists if the curve Γ is smooth or piecewise-smooth, and the density f satisfies the Hölder condition with an exponent $\nu\in(0;1]$. We denote the class of all that functions $H_\nu(\Gamma)$.

If $f\in H_\nu(\Gamma)$ and Γ is smooth, then the function $\mathcal{S}_\Gamma f(t)$ is continuous and satisfies the Hölder condition with the same exponent ν for $\nu<1$, and with arbitrarily close to unit exponent for $\nu=1$. But this result is not valid at the corners of a piecewise-smooth curve. For example, if Γ is boundary of a square with counterclockwise circuit and $f\equiv 1$, then the function $\mathcal{S}_\Gamma f$ equals 1 at all points of Γ excluding the vertices, where it is equal to $3/2$.

In this connection there arises another definition of the Cauchy singular integral:

$$\mathcal{S}^*_\Gamma f(t) := f(t)+\frac{1}{\pi i}\int\limits_\Gamma\frac{f(\tau)-f(t)}{\tau-t}d\tau, \quad t\in\Gamma, \tag{2}$$

where the integral is understood as improper. At the points of smoothness of the curve the functions $\mathcal{S}_\Gamma f(t)$ and $\mathcal{S}^*_\Gamma f(t)$ coincide, but at the corners the second one keeps continuity. For instancee, $\mathcal{S}^*_\Gamma 1(t)=1$ at all points of the boundary of the square, including its vertices.

The formula (2) keeps validity for certain non-smooth rectifiable curves. For instance, the integral (2) converges if $f\in H_\nu(\Gamma)$ for $\nu>0$, and rectifiable curve Γ is AD-regular, i.e. the sum of lengths of its arcs covered by any disc of radius r does not exceed Cr, where positive constant C does not depend on the center of the disk and on r.

There exists one more approach to the definition of the singular integral. Let us consider the jump problem, i.e., the problem on reconstruction of analytic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that it vanishes at the point at infinity, and has at every point $t \in \Gamma$ continuous limit values $\Phi^{\pm}(t)$ from domains $D^{\pm}$ satisfying relation

$$\Phi^{+}(t) - \Phi^{-}(t) = f(t), \tag{3}$$

where f ia a given function defined on Γ. If curve Γ is smooth and $f \in H_{\nu}(\Gamma)$, then a unique solution of this problem (see [1–3]) is the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(t)\, dt}{t - z}, \tag{4}$$

and $\Phi^{\pm}(t) = \frac{1}{2}(\pm f(t) + \mathcal{S}_{\Gamma} f(t))$. For piecewise-smooth curves the last formula keeps its validity at the angular points if we replace there $\mathcal{S}_{\Gamma} f(t)$ by $\mathcal{S}^{*}_{\Gamma} f(t)$. A solution of the jump problem is unique for any rectifiable curve. This fact follows from the following Painleve theorem (see [6]): if a function is continuous in domain D and analytic in $D \setminus \Gamma$, where a curve Γ is rectifiable, then this function is analytic in D, too. Therefore, we are able to define an analog of the singular integral for non-smooth rectifiable curve as sum

$$\mathcal{S}^{\Gamma} f(t) = \Phi^{+}(t) + \Phi^{-}(t), \quad t \in \Gamma, \tag{5}$$

where Φ is a unique solution of the jump problem (3), if it exists.

This definition is expandable on non-rectifiable curves of the following class. Let a curve Γ contain a finite set of points E such that for any its neighborhood $N(E)$ the difference $\Gamma \setminus N(E)$ is the union of a finite number of rectifiable arcs. We call that curve E^{c}-rectifiable. For example, the arc

$$\{z = x + iy : -1 \le x \le +1, y = x \sin x^{-p}\}$$

is 0^{c}-rectifiable, but it is not rectifiable for $p \ge 1$. Clearly, the Painleve theorem is valid for E^{c}-rectifiable curves, what enables us to apply the last definition of the Cauchy integral to that curves.

All three definitions lead to the same result at the points of smoothness of the curve. Here we study the singular integral (5) for non-smooth and non-rectifiable curves Γ.

Let us describe our class of curves. Let Γ' and Γ be simple closed curves bounding finite domains D' and D relatively. The set $D' \triangle \overline{D}$ consists of a finite or infinite set of mutually disjoint domains Δ_j, $j = \pm 1, \pm 2, \dots$. Each of these domains has signature s_j equaling $+1$ for $\Delta_j \subset D'$, and -1 for $\Delta_j \subset D$. We say that the curve Γ is a perturbation of Γ' of type $A(t_0)$ if
(1) all boundaries γ_j of domains Δ_j are simple piecewise-smooth curves;

(2) the family of domains Δ_j is infinite and has a unique condensation point t_0.

Theorem 1. *Let a t_0^c-rectifiable curve Γ be a perturbation of type $A(t_0)$ of a smooth curve Γ', and $f \in H_\nu(\Gamma)$. Then singular integral $\mathcal{S}^\Gamma f$ exists at all points of the curve including the condensation points t_0 if the series*

$$\sum_{j=1}^{+\infty} \iint\limits_{\Delta_j} \frac{dx\,dy}{\operatorname{dist}^q(x+iy;\Gamma)}, \quad \sum_{j=-1}^{-\infty} \iint\limits_{\Delta_j} \frac{dx\,dy}{\operatorname{dist}^q(x+iy;\Gamma)} \tag{6}$$

converge for certain $q > 2(1-\nu)$.

Note that the integral

$$\iint\limits_{\Delta_j} \frac{dx\,dy}{\operatorname{dist}^q(x+iy;\Gamma)}$$

diverges for $q \geq 1$ if the set $\Gamma \cap \gamma_j$ contains a continuum. Therefore, the assumptions of Theorem imply the inequality $\nu > 1/2$.

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НАИЛУЧШИЕ РАВНОМЕРНЫЕ РАЦИОНАЛЬНЫЕ ПРИБЛИЖЕНИЯ ПРЕОБРАЗОВАНИЙ КОШИ С ЛОГАРИФМИЧЕСКИМИ И СТЕПЕННО-ЛОГАРИФМИЧЕСКИМИ ОСОБЕННОСТЯМИ МЕР

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