UDC 517.984

ON RECOVERING STURM – LIOUVILLE OPERATORS WITH FROZEN ARGUMENT [1]

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1. Introduction. Inverse problems of spectral analysis consist in recovering operators from given their spectral characteristics. The greatest success in the inverse problem theory has been achieved for the Sturm – Liouville operator and afterwards for higher-order differential operators (see [1, 2] and references therein). Inverse problems for differential operators with deviating argument and other classes of non-local operators because of their difficulty were studied insufficiently. Some aspects of inverse spectral theory for differential operators with fixed argument were investigated in [3].

Fix $k \in \mathbb{N}$ and consider the boundary value problem $L = L(q(x))$:

$$\ell y := -y'' + q(x)y\Big(\frac{\pi}{k}\Big) = \lambda y, \quad 0 < x < \pi. \tag{1.1}$$

$$y(0) = y(\pi) = 0. \tag{1.2}$$

Here λ is the spectral parameter, $q(x)$ is a complex-valued function and $q(x) \in L_2(0, \pi)$. We refer to ℓ as to the Sturm-Liouville operator with frozen (fixed) argument. Let $\{\lambda_n\}_{n\geq 1}$ be the spectrum of L. Consider the following inverse problem.

Inverse Problem 1. Given $\{\lambda_n\}_{n\geq 1}$, find $q(x)$.

We obtain necessary and sufficient conditions for solvability of Inverse Problem 1 and prove uniqueness of its solution in the class of symmetric potentials. A constructive procedure for solving the inverse problem is provided.

2. Characteristic function. Properties of the spectrum. Let $C(x, \lambda)$, $S(x, \lambda)$ be solutions of equation (1.1) under the initial conditions

$$C\Big(\frac{\pi}{k}, \lambda\Big) = S'\Big(\frac{\pi}{k}, \lambda\Big) = 1; \quad S\Big(\frac{\pi}{k}, \lambda\Big) = C'\Big(\frac{\pi}{k}, \lambda\Big) = 0.$$

Eigenvalues of L coincide with the zeros of its characteristic function

$$\Delta(\lambda) := S(\pi, \lambda)C(0, \lambda) - S(0, \lambda)C(\pi, \lambda).$$

[1] This research was supported in part by RFBR (Grants no. 15-01-04864, no. 16-01-00015) and by the Ministry of Education and Science of RF (Grant no. 1.1436.2014K).

Lemma 1. *Characteristic function of the boundary problem L has the form*

$$\Delta(\lambda) = \frac{\sin\rho\pi}{\rho} + \int_0^\pi W(t)\frac{\cos\rho t}{\rho^2}\,dt, \quad W(t) \in L_2(0,\pi). \tag{2.1}$$

Moreover,

$$W(t) = \frac{1}{2}\begin{cases} q\Big(\pi\frac{k-1}{k}+t\Big) + q\Big(\pi\frac{k-1}{k}-t\Big), & x \in \Big(0,\frac{\pi}{k}\Big), \\ q\Big(\pi\frac{k-1}{k}-t\Big) - q\Big(\pi\frac{k+1}{k}-t\Big), & x \in \Big(\frac{\pi}{k},\frac{\pi(k-1)}{k}\Big), \\ -q\Big(\pi\frac{k+1}{k}-t\Big) - q\Big(t-\pi\frac{k-1}{k}\Big), & x \in \Big(\frac{\pi(k-1)}{k},\pi\Big). \end{cases}$$

Using the properties of $W(t)$, we arrive at the following assertion.

Lemma 2. *The function $\Delta(\lambda)$ has the form*

$$\Delta(\lambda) = \frac{\sin\rho\pi}{\rho} + \sin\frac{\pi\rho}{k}\int_0^{\pi\frac{k-1}{k}} V(t)\frac{\sin\rho t}{\rho^2}\,dt, \quad V(t) \in L_2\Big(0,\pi\frac{k-1}{k}\Big). \tag{2.2}$$

Theorem 1. *The boundary value problem L has a countable set of eigenvalues $\{\lambda_n\}_{n\geq 1}$ of the form*

$$\lambda_n = \Big(n + \frac{\kappa_n}{n}\Big)^2, \quad \{\kappa_n\} \in l_2. \tag{2.3}$$

Moreover, a k-th part of eigenvalues degenerate, in the following sense:

$$\lambda_{kn} = (kn)^2, \quad n \in \mathbb{N}. \tag{2.4}$$

Lemma 3. *The specification of the spectrum $\{\lambda_n\}_{n\geq 1}$ uniquely determines the characteristic function by the formula*

$$\Delta(\lambda) = \pi\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2}. \tag{2.5}$$

Lemma 4. *Let arbitrary complex numbers λ_n, $n \geq 1$, of the form* (2.3)*, satisfying* (2.4)*, be given. Then the function $\Delta(\lambda)$ determined by* (2.5) *has the form* (2.2).

3. Solution of the inverse problem. The following theorem gives necessary and sufficient conditions for solvability of Inverse Problem1.

Theorem 2. *For arbitrary sequence of complex numbers* $\{\lambda_n\}_{n\geq 1}$ *to be the spectrum for a certain boundary value problem* $L = L(q(x))$ *it is necessary and sufficient to satisfy* (2.3), (2.4).

In the class of potentials that are symmetric with respect to the point $x = \pi/k$ the following uniqueness theorem is valid.

Theorem 3. *Let* $q(x) = q(2\pi/k - x),\ x \in (0, \pi/k)$, *then the specification of spectrum uniquely determines the potential* $q(x)$.

The solution of Inverse Problem 1 in the class of symmetric potentials can be found by the following algorithm.

Algorithm 1. Let the spectrum $\{\lambda_n\}_{n\geq 1}$ be given.

1. Construct $\Delta(\lambda)$ by formula (2.5);
2. Find $W(x)$, $x \in (0, \pi)$ from (2.1) inverting the Fourier transform;
3. Construct $q(x),\ x \in (0, 2\pi/k)$ using

$$q(x) = \begin{cases} -W\Big(\pi\dfrac{k-1}{k} + x\Big), & x \in \Big(0, \dfrac{\pi}{k}\Big), \\ -W\Big(\pi\dfrac{k+1}{k} - x\Big), & x \in \Big(\dfrac{\pi}{k}, 2\dfrac{\pi}{k}\Big); \end{cases}$$

For $\nu = \overline{2, k-2}$ repeat the following step:

4. If $q(x)$ on $(0, \nu\pi/k)$ is calculated, then find $q(x)$ on $\left(\frac{\nu\pi}{k}, \frac{(\nu+2)\pi}{k}\right)$ by the formula

$$q(x) = q\Big(x - 2\frac{\pi}{k}\Big) - 2W\Big(\pi\frac{k+1}{k} - x\Big). \tag{3.1}$$

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