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SUBINVERTIBILITY OF COMPACT-VALUED SUBLINEAR OPERATORS[1]

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In the problems of modern nonsmooth analysis and nonsmooth optimization, the multivalued sublinear operators plays, as is known, ever more important role (see, e.g., [1–5]). In particular, the concepts of the subdifferential and the subsmoothness, which were researched in the series of our works [6–8], are jointly connected to multivalued sublinear operators, that take convex compact values.

The present work represents an outline of such theory. We describe the compact-valued sublinear operators by means of the packets of single-valued so called basic selectors. This makes it possible to introduce a concept of the multivalued invertibility through the concept of the corresponding selectors. The construction and the properties of the invertible multioperators are described explicitly. Special attention is given to the question on extremal points of selector representation and corresponding application of Krein – Milman theorem.

1. Preliminaries. Sublinear K-operators and their simplest properties

Definition 1. Let F be a real normed space. Denote by F_K the convex cone consisting of all non-empty convex compacts from F, equipped with element-wise addition, non-negative scalar multiplication and the cone-norm:

$$\|C\| = \sup_{y\in C} \|y\| \quad (C \subset F).$$

Definition 2. Let E and F be the real normed spaces, $A:\ E \to F_K$. Say that A is a *sublinear K-operator* if the following properties

(i) $A(h+k) \subset Ah + Ak$ (subadditivity);

(ii) $A(\lambda h) = \lambda \cdot Ah$ $(\lambda \geq 0)$ (positive homogeneity);

are satisfied. A cone-norm for the K-operator is introduced in the standard way:

$$\|A\| = \sup_{\|h\|\leq 1} \|Ah\|.$$

Say that the K–operator A is *bounded* if $\|A\| < \infty$. The normed cone of all bounded K-operators $A:\ E \to E_K$ denote by $L_K(E;F)$.

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For our purposes it is more appropriate to use *totally homogeneous* K-operators ($A(\lambda h) = \lambda \cdot Ah$ ($\forall\, \lambda \in \mathbf{R}$)).

2. Constructing of the packet of basic selectors for a given K-operator

In what follows, $H = \{h_i\}_{i\in I}$ is some fixed normed Hamel basis in some real Banach space E.

Definition 3. Let $A \in L_K(E;F)$. Choose an arbitrary element $a_i \in Ah_i$ for each $i \in I$ and set

$$A^s h_i = a_i \, (\forall\, i \in I), \quad A^s h = \sum_{k=1}^{n} \lambda_k a_{i_k} \quad \left(h = \sum_{k=1}^{n} \lambda_k h_{i_k} \in E \right).$$

Let us call the set $A_K = \{A^s\}$ the *packet of basic selectors* (or *s-representation*) of the sublinear K-operator A.

Note that s-representation depends on the choice of the Hamel basis H in E. First, explain that all basic selectors are linear continuous operators.

Theorem 1. *Let E and F be the real Banach spaces, H be some Hamel basis in E and $A \in L_K(E;F)$. Then for every selector $A^s \in A_K$ the following properties:*

$$A^s \in L(E;F); \quad \|A\| \le \sup_{A^s \in A_K} \|A^s\| \le C \cdot \|A\|; \tag{1}$$

are valid. Here the constant $C = C(H)$ from the right in (1) *depends only on the choice of some Hamel basis H in E.*

Remark. It is possible to identify the packet of basic selectors $A_K = \{A^s\}$ and the K-operator $A_K h = \{A^s h |\, A^s \in A_K\}$. Then the estimate (1) can be rewritten in the form of norm equivalence for the K-operators A and A_K:

$$\|A\| \le \|A_K\| \le C_H \cdot \|A\| \quad (\forall\, A \in L_K(E;F)), \tag{2}$$

where the constant C_H depends only on the choice of Hamel basis H. In addition, $Ah \subset A_K h$ ($\forall\, h \in E$) and the correspondence $A \mapsto A_K$ is sublinear. Note that, under such representation, $A_K \in L_K(E;F)$.

Theorem 2. *For every K-operator $A \in L_K(E;F)$ its s-representation A_K is a sublinear bounded K-operator, as well.*

Theorem 3. *For every K-operator $A_K \in L_K(E;F)$ its packet of basic selectors A_K is a convex compact in $L(E;F)$.*

Finally, if the Banach space E possesses a topological basis then, it is possible to describe the K-operator A_K by means of its values on the basis in E.

Corollary. *Let a real Banach space have a topological basis* $\{h_n\}_{n=1}^{\infty}$, $A \in L_K(E;F)$. *Then, under a suitable choice of Hamel basis in* E, *the following equality holds:*

$$(h = \sum_{n=1}^{\infty} \lambda_n h_n \in E) \Rightarrow \left((A_K h = \sum_{n=1}^{\infty} \lambda_n \cdot A h_n \,\middle|\, \mathrm{dist}_H \right).$$

Here we denote by $dist_H$ *the Hausdorff metric in the set of all compacts from* E.

3. K-invertibility of K-operators

Definition 4. Say that the K-operator A is K-*invertible* if $A_K \subset \subset \mathrm{Isom}(E;F)$. In this case, introduce K-*inverse* K-operator A_K^{-1} as follows:

$$A_K^{-1} = \overline{\mathrm{co}} \left\{ (A^s)^{-1} | \, A^s \in A_K \right\} \quad \left(A_K^{-1} k = \left\{ B^\sigma k | \, B^\sigma \in A_K^{-1} \right\} \right).$$

The set of the all K-invertible K-operators $A : E \to F_K$ denote by $\mathrm{Isom}_K(E;F)$.

Consider some properties of the K-invertible K-operators.

Theorem 4. *If the* K*-operator* A *is* K*-invertible then* A_K^{-1} *forms a convex compact in* $L(E;F)$.

Theorem 5. *If the* K*-operator* A *is* K*-invertible then*

$$I_E h \in [A_K^{-1} \cdot A_K]h \ (h \in E); \quad I_F k \in [A_K \cdot A_K^{-1}]k \ (k \in F). \tag{3}$$

Next, let's explain the structure of the operator that is K-inverse to the K-composition (see [8]).

Theorem 6. *If* $A \in Isom_K(E;F)$, $B \in Isom_K(F;G)$, *then*

$$[B \cdot A]_K^{-1}(l) \subset [B_K \cdot A_K]_K^{-1}(l) \subset [A_K^{-1} \cdot B_K^{-1}](l) \quad (\forall \, l \in G).$$

At last, let's explain the question on the repeated K-invertibility.

Theorem 7. *If* $A \in Isom_K(E;F)$ *then* $A_K^{-1} \in Isom_K(F;E)$, *in addition*

$$A_K h \subset (A_K^{-1})_K^{-1} \cdot h \quad (\forall \, h \in E). \tag{3}$$

Compactness and convexity of the packets A_K and A_K^{-1} lead to the actual problem of describing the extremal points of these sets.

In what follows, we denote by $\mathrm{Extr}(C)$ the set of all extremal points of C; here C is a convex compact set either from F, or from $L(E;F)$, E and F are real Banach spaces, $H = \{h_i\}_{i \in I}$ is some Hamel basis in E. First, let's obtain a description of $\mathrm{Extr}(A_K)$.

Theorem 8. *Let $A \in L_K(E;F)$, $A_K = \{A^s\}$ be its s-representation. Then*

$$(A^s \in \mathrm{Extr}(A_K)) \Leftrightarrow (\forall\, h_i \in H :\ A^s h_i \in \mathrm{Extr}(Ah_i)). \tag{4}$$

Now, let's pass to the case of the K-invertible K-operator $A \in \mathrm{Isom}_K(E;F)$. Thus, $A_K \in \mathrm{Isom}_K(E;F)$, $A_K^{-1} \in \mathrm{Isom}_K(F;E)$.

Theorem 9. *If $A \in Isom(E;F)$ then the following inclusion*

$$(Extr\, A_K)^{-1} \subset Extr(A_K^{-1}). \tag{5}$$

takes place.

Corollary. *Under the conditions of Theorem 8 the following inclusions*

$$\overline{co}\,(Extr\, A_K)^{-1} \subset A_K^{-1}; \qquad A_K \subset \overline{co}\,(Extr\, A_K^{-1})^{-1};$$

take place.

Now let's consider a question on sufficient condition of K-invertibility, namely, on K-analog of the known von Neumann theorem.

Theorem 10. *Let $A \in L_K(E)$. If $A = I - B$, where $\|B_K\| < 1$, then A is K-invertible. Moreover, the following estimate*

$$A_K^{-1} h \subset (I + \sum_{n=1}^{\infty} B_K^n) h \quad (\forall\, h \in E). \tag{6}$$

takes place. Here in (6) the power of the K-operator is meant with respect to the K-product (see [8]), and the convergence of the power series in (6) is meant with respect to the cone-norm in $L_K(E)$.

By applying the Krein – Milman theorem, now it is easy to obtain

Theorem 11. *Let, under the conditions of Theorem 10, the inequality*

$$\|A_e - I\| \leq 1 - \varepsilon$$

holds for all extremal points $A_e \in Extr\, A_K$. Then A is K-invertible.

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GEOMETRIC PROPERTIES OF WRIGHT FUNCTION

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The Wright function is define by

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}, \quad \lambda > -1,\ \mu \in \mathbb{C}. \tag{1}$$

This function was studied in connection with the partitions of the natural numbers [1]. For $\lambda > -1$, the Wright function is an entire function of z and has been used widely in the asymptotic theory of partitions, Mikusinski operational calculus, integral transforms and in fractional differential equations [2]. For $\lambda = 1$, $\mu = \nu + 1\,(\nu > -1)$, the function $W_{\lambda,\mu}$ can be represented in terms of the Bessel functions. Also, Wright function generalizes various functions like Array function, Whittakar function, (Wright-type) entire auxiliary functions etc.

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ which are also univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is called starlike, denoted by $\mathcal{S}^*$ if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0,1]$. A function $f \in \mathcal{A}$ is called starlike function, if $\mathrm{Re}\,(zf'(z)/f(z))\,hm > 0$, $z \in \mathbb{D}$. Further, we denote by $\widetilde{\mathcal{S}}^*(\alpha)$, $0 < \alpha \leq 1$, the class of strongly starlike functions of order α, which is defined by

$$\widetilde{\mathcal{S}}^*(\alpha) = \left\{ f \in \mathcal{A} : \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \right\}. \tag{3}$$

Note that $\widetilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$. An analytic function f is called close-to-convex in $\mathbb{D}$, if complement of $f(\mathbb{D})$ can be written as the union of non-intersecting